TOPOLOGICAL METHODS IN SECOND ORDER ARITHMETIC

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CONTENTS

PREFACE

The purpose of this thesis is to show the use of topology in mathematical logic.

It is well known, through the work of H. Rasiowa and R. Sikorski, that Intuitionistic first order predicate calculus with equality has a very natural interpretation in terms of the lattice of open sets of a topological space. The problem was to show that the usual Intuitionistic second order arithmetic gives such a topological model.

In the first two chapters we give the topology required for later work and a brief discussion on Intuitionistic mathematics. In the topology section the emphasis is on the Baire Space as this is the most natural space in which to interpret second order arithmetic. The discussion on Intuitionistic mathematics gives an Intuitionistic viewpoint but concentrates on giving, where possible, Classical justification for Intuitionistic assumptions and theorems.

Chapter Three is a brief description of Ra owa's and Sikorski's work showing that Intuitionistic first order predicate calculus with equality has topological models.

Chapter Four gives a model for Intuitionistic second order arithmetic which is due to J.R. Moschovakis and is an adaption of topological models for Intuitionistic analysis given by D. Scott.

Chapter Five shows a second use of topology in mathematical logic. It deals with the topological approach of forcing due to G. Takeuti and C. Ryll-Nardzewski. The central idea is that by using the Baire Category Theorem we can show the existence of generic models without giving a constructive proof.

Chapter One (except 1.2.9 and 1.3.7) and the proof of 2.5.7 are taken from lectures given by R.A. Bull to a 1976 Honours III Logic class. The details of Chapter Five were worked out with the help of R.A. Bull.

Acknowledgements.

I thank Dr R.A. Bull for his supervision, which kept me on the right path, and for undertaking the painstaking job of proof reading the typewritten script. Finally I thank Mrs Climo for the difficult task of typewriting this thesis.

1. BAIRE SPACE

1.1. THE TOPOLOGY OF BAIRE SPACE.

1.1.1. Remark. We shall consider an Intuitionistic second order arithmetic which has added variables for functions. Since functions from N (the natural numbers) into N are just sequences, then to study models of second order arithmetic we need to know some of the properties of sequences.

There are three reasons for looking at the Baire Space topology. Firstly we shall later show that the models of Intuitionistic logic are lattices of open sets of given topological spaces and in our context a natural topological space to consider is the Baire Space. Secondly knowing something of the Baire Space topology shows us that when an Intuitionist describes $\forall\alpha\exists xX(\alpha,x) \to \forall\alpha\exists x\exists y\forall\beta(\bar{\alpha}y = \bar{\beta}y \to X(\beta,x))$ as a continuity axiom, he is not pulling this term out of the hat, but is linking it with the notion of continuity in topology. Thirdly, via Baire Space topology, we can give classical proofs for assumptions made by intuitionists which they justify on other grounds.

It must be stressed that Intuitionism did not develop as some extension of Classical mathematics as the above may suggest. Rather Brouwer attempted to create a new mathematics which is constructivist and so he justified his assumptions on philosophical grounds. It was only when people attempted to formalize Intuitionism that semantic interpretations such as topological models were developed.

In this thesis we have substituted for any historical approach an approach summarized as 'there is a topological model; let us see how Intuitionistic second order arithmetic is interpreted in this model'. Thus for the most part we avoid a detailed philosophical discussion on the constructive content of intuitionism.

<u>1.1.2. Definitions</u>. We use x, y, z,... for non-negative integers (nnis) and N for the set of nnis.

We have functions $+$, $\times$ and $\dot{-}$, $^{+}$, $^{-}$, on nnis. The first two have their usual definition, the last three are defined as follows:

$$x \dot{-} y =_{df} x - y \quad \text{iff } x \geqslant y$$
$$x \dot{-} y =_{df} 0 \quad \text{iff } x < y$$
$$x^{+} =_{df} x + 1 \qquad x^{-} =_{df} x \dot{-} 1$$

We have finite sequences of nnis $< x_0, \ldots x_r >$ including the empty sequences $< >$. We use m, n for these, and if m is $< x_0 \ldots x_r >$ then we write ms for x_s, for each s with $0 \leqslant s \leqslant r$.

We define the function lh on finite sequences as follows: for each $m = < x_0 \ldots x_1 >$,$lh< > =_{df} 0 \quad lh\ m = r^{+}$.

We define a concatenation function * on finite sequences as follows:

for each $< x_0 \ldots x_r >, < y_0 \ldots y_s >, < x_0 \ldots x_r > * < y_0 \ldots y_s > =_{df}$

$$< x_0 \ldots x_r y_0 \ldots y_s > .$$

We define a 'tail function' tl on finite sequences as follows:

for each $m = < x_0 \ldots x_r >$, $tl\ m =_{df} < x_1 \ldots x_r >$

i.e. $< x_0 > * \ tl\ m = m$.

We define a relation $\leqslant$ on finite sequences as follows: for each $<x_0 \dots x_r>, <y_0 \dots y_s>, <x_0 \dots x_r> \leqslant <y_0 \dots y_s> =_{df} r \leqslant s$ and $<x_0 \dots x_r> = <y_0 \dots y_r>$. Note that this relation is reflexive, antisymmetric and transitive, i.e. a partial-ordering. $<$ is defined in the obvious way,

$m < n =_{df} m \leqslant n$ and not $m = n$.

We use α, β, γ,... for sequences of nnis and N^N for the set of sequences. We define a function $\bar{\alpha}$ from N into the set of finite sequences as follows:

$\bar{\alpha}0 = < >$ $\bar{\alpha}r^+ = <\alpha 0 \alpha 1 \dots \alpha r>$ for each $\alpha \in N^N$, each $r \in N$.

Given a finite sequence m we write {m} for the set of sequences α such that $\bar{\alpha}(\text{lhm}) = m$ i.e. $\{m\} =_{df} \{\alpha : \bar{\alpha}(\text{lhm}) = m\}$. $\{<>\}$ is N^N.

Given a finite sequence m and a sequence α, $m * \alpha$ is the sequence defined as follows:

$$(m * \alpha) r = mr \qquad \text{for } r \in N, \quad r \leqslant (\text{lhm})$$

$$(m * \alpha)((\text{lhm}) + s) = \alpha s \qquad \text{for } s \in N.$$

<u>1.1.3. Remarks.</u> We now ask what topology is to be put on N^N. The natural topology is the induced product topology (with the discrete topology on N). The subbasic open sets of this topology are of the form $\{\alpha : \alpha r = x\}$, for some $r, x \in N$, so that the basic open sets are of the form $\{\alpha : \alpha r_1 = x_1$ and ... and $\alpha r_n = x_n\}$, for some $r_1, \dots, r_n, x_1, \dots, x_n \in N$.

Alternatively we may view this topology as the one whose basic open sets are {m} for any finite sequence m. This is easily seen by noting that for any m with $m = <x_0 \dots x_r>$, $\{m\} = \{\alpha : \alpha 0 = x_0$ and ... and $\alpha r = x_r\}$, which is a basic open set of the product topology, and for any

$\{\alpha : \alpha r_1 = x_{r_1}$ and ... and $\alpha r_n = x_{r_n}\}$ this equals

$\{\{\alpha : \alpha 0 = x_0$ and ... and $\alpha(r_1 - 1) = x_{r_1 - 1}$ and $\alpha r_1 = x_{r_1}$

and $\alpha(r_1 + 1) = x_{r_1+1}$ and ... and $\alpha(r_n - 1) = x_{r_n - 1}$

and $\alpha r_n = x_{r_n}\} : x_0, \ldots x_{r_1-1}\ x_{r_1+1}, \ldots x_{r_n-1} \in N\}$

which is the union of basic open sets of the form $\{m\}$ for some m.

1.1.4. Definition. This topology whose basic open sets are $\{m\}$ for some finite sequence m, is called the Baire Space topology.

1.1.5. Proposition. The Baire Space has a countable basis.

Proof. In view of the definition of the Baire Space topology we need only show that the set of finite sequences is countable.

To each ordered pair $\langle x,y\rangle$ of nnis assign a nni (x,y) by $(x,y) = \frac{1}{2}[(x+y)^2 + 3x + y]$. By noting that

$(x,y) = \left(\sum_{n=1}^{x+y} n\right) + x$ we can easily show that this assignment from $N \times N$ into N is 1-1 and onto.

To each finite sequence assign an nni in the following way. To $\langle\ \rangle$ assign 0, to $\langle x_0\rangle$ assign $(0, x_0)^+$, for each $r \geqslant 1$ to $\langle x_0 \ldots x_r\rangle$ assign $(r, (x_0, (x_1, \ldots (x_{r-1}, x_r), \ldots)))^+$.

In view of the 1-1, onto assignment of (x,y) and the 1-1 assignment of $^+$ from N onto $N - \{0\}$, and the 1-1, onto assignment of 0 to $\langle\ \rangle$, this assignment being the composition of those assignments is 1-1 and onto.

<u>1.1.6. Proposition.</u> Members of the countable base for the Baire Space are clopen.

<u>Proof.</u> We need only show that for each $\{m\}$ in the countable basis $-\{m\}$ is open.

$\alpha \in -\{m\}$ implies $\bar{\alpha}(\mathrm{lh}m) \neq m$ and $\alpha \in \{\bar{\alpha}(\mathrm{lh}m)\}$

implies $\bar{\alpha}(\mathrm{lh}m) \neq m$ and $\mathrm{lh}(\bar{\alpha}(\mathrm{lh}m)) = \mathrm{lh}m$

implies not $\bar{\alpha}(\mathrm{lh}m) \leqslant m$ and not $m \leqslant \bar{\alpha}(\mathrm{lh}m)$

implies for some $n, \alpha \in \{n\}$ and not $n \leqslant m$ and not $n \leqslant m$.

For some $n, \alpha \in \{n\}$ and not $n \leqslant m$ and not $n \leqslant m$ and $\alpha \notin -\{m\}$

implies $\alpha \in \{m\}$

implies for each n with $\{m\} \cap \{n\} = \phi, \alpha \notin \{n\}$

implies for each n with not $m \leqslant n$ and not $n \leqslant m$, $\alpha \notin \{n\}$

This contradicts the assumption, for some n, not $n \leqslant m$ and not $m \leqslant m$ and $\alpha \in \{n\}$. Hence, for some n, $\alpha \in \{n\}$ and not $n \leqslant m$ and not $m \leqslant m$ implies $\alpha \in -\{m\}$.

Therefore $-\{m\} = \cup \{\{\{n\} : \text{not } n \leqslant m \text{ and not } m \leqslant n\} : n \in N\}$ which is open.

<u>1.1.7. Proposition.</u> Each open set in the Baire Space topology is the union of countably many disjoint clopen sets.

<u>Proof.</u> In view of 1.1.5, 1.1.6 we need only show that each open set is the union of disjoint basis neighbourhoods.

Let A be any open set and let N be the set of basis neighbourhoods which are contained by A so that $A = \cup N$.

Let M be the set of members $\{n\}$ of N for which

$\{m\} \in M$ iff $\{m\} \in N$ and for each n, if $n \leqslant m$ and $\{n\} \in N$ then $n = m$.

Because for finite sequences m and n either $\{m\} \subseteq \{n\}$ or $\{m\} \supseteq \{n\}$ or $\{m\} \cap \{n\} = \phi$, then for each distinct $\{m\}$, $\{n\} \in M$, not $\{m\} \subseteq \{n\}$ and not $\{n\} \subseteq \{m\}$; otherwise either n is not minimal or m is not minimal. Hence M is a set of disjoint neighbourhoods.

For each $\alpha \in A$, for some $\{n\} \in N$, $\alpha \in \{n\}$. Since there are lhn finite sequences m such that $m \leqslant n$, there is a least finite sequence m under $\leqslant$ with $m \leqslant n$ and $\{m\} \in N$. Call this m_0.

m_0 is minimal under $\leqslant$ because

$m \leqslant m_0$ and $\{m\} \in M$ implies $m \leqslant n$ and $\{m\} \in N$ and $m \leqslant m_0$
implies $m = m_0$ by choice of m_0 as the least such m.

Therefore $\{m_0\} \in M$ and $\alpha \in \{n\} \subseteq \{m_0\}$. Hence $\alpha \in \cup M$.
Therefore

$$A \subseteq \cup M \subseteq \cup N \subseteq A \quad \text{i.e. } A = \cup M.$$

<u>1.1.8. Remark</u>. The next few propositions tell us certain properties of the Baire Space that, as well as placing this space within the general scheme of topological spaces, are needed to prove results about models in Chapter Three.

<u>1.1.9. Definition</u>. Let X be a topological space, $x \in X$. The component $C(x)$ of point $x =_{df}$ the union of all connected sets containing x : i.e. it is the maximal connected set containing x.

X is totally disconnected $=_{df}$ for each $x \in X$, $C(x) = \{x\}$.

<u>1.1.10. Remark.</u> A more convenient formulation of totally disconnectedness is: for each distinct $x, y \in X$, there exist disjoint open A and B with $x \in A$, $y \in B$, $X = A \cup B$. This is equivalent to the definition in 1.1.9 as is seen by:

for each $x \in X$ $C(x) = \{x\}$ implies X is not connected

implies there exist disjoint open A, B with $X = A \cup B$.

If $x \in A$ and $y \in B$ then the proof is finished.

If $x, y \in A$, say, then by the maximality of $C(x)$, $C(y)$, A is not connected

implies there exist disjoint open (in the subspace topology) A_1, B_1 with $A = A_1 \cup B_1$

A open implies there exist disjoint open (in the topology) A_1, B_1 with $A = A_1 \cup B_1$.

If $x \in A_1$, $y \in B_1$ then let the disjoint open sets be A_1, $B_1 \cup B$.

If $x, y \in A_1$, say, then by the maximality of $C(x)$, $C(y)$, A_1 is not connected. We repeat the argument until eventually either $\{x\}$, $\{y\}$ are open or there exist A_n, B_n with $x \in A_n$, $y \in B_n$. In the former case since closure of a connected set is connected then by the maximality of $C(x)$, $\{x\}$ is closed. Take the open sets to be $\{x\}$, $X - \{x\}$. In the latter case take the open sets to be A_n, $B_n \cup B_{n-1} \ldots \cup B_1 \cup B$.

For each distinct $x, y \in X$ there exist disjoint open A, B with $x \in A$, $y \in B$, $X = A \cup B$, and suppose there is a y with $C(y) \neq \{y\}$. Then this

implies there is an $x \neq y \in C(y)$ and there exist disjoint open A, B with $x \in A$, $y \in B$, $X = A \cup B$

implies there exist disjoint open (in the subspace topology)

$C(y) \cap A$, $C(y) \cap B$ with $x \in C(y) \cap A$, $y \in C(y) \cap B$ and

$C(y) = (C(y) \cap A) \cup (C(y) \cap B)$

implies $C(y)$ is not connected. Contradiction.

Note that any totally disconnected space is Hausdorff.

1.1.11. Proposition. The Baire Space is totally disconnected.

Proof. We need only show that given distinct sequences α, β there is a clopen set with $\alpha \in A$ and $\beta \notin A$.

Let α, β be distinct sequences, let r be the smallest nni with $\alpha r \neq \beta r$. The set $\{\bar{\alpha}r^{+}\}$ is clopen by 1.1.6 and $\alpha \in \{\bar{\alpha}r^{+}\}$ and $\beta \notin \{\bar{\alpha}r^{+}\}$ since $(\bar{\alpha}r^{+})r = \alpha r \neq \beta r$.

1.1.12. Proposition. The Baire Space is a complete metric space.

Proof. Define a metric d on the Baire Space as follows:

$d(\alpha, \beta) =_{df} 0$ iff $\alpha = \beta$ $\quad d(\alpha, \beta) =_{df} \frac{1}{r^{+}}$ iff r is the least nni with $\alpha r \neq \beta r$.

It is easy enough to show that d is a metric; the hardest part as always is to show the triangle property.

Consider sequences $\alpha_0, \alpha_1, \ldots$ such that for each r, for some p_r, for each p, q with $p_r \leqslant p$, $p_r \leqslant q$, $d(\alpha_p, \alpha_q) < \frac{1}{r^{+}}$.

This means the least nni s with $\alpha_p s \neq \alpha_q s$ satisfies $d(\alpha_p, \alpha_q) = \frac{1}{s^{+}} < \frac{1}{r^{+}}$.

Thus $r < s$ and therefore $\bar{\alpha}_p r^{+} = \bar{\alpha}_q r^{+}$.

Hence for each r, for some p_r, for each p and q, $p_r \leqslant p$ and $p_r \leqslant q$ implies $\bar{\alpha}_p r^{+} = \bar{\alpha}_q r^{+}$

For each r, take p_r to be the least nni which satisfies this statement so that $p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ and hence $\bar{\alpha}_{p_0}0^+ = \bar{\alpha}_{p_1}0^+$, $\bar{\alpha}_{p_1}1^+ = \bar{\alpha}_{p_2}1^+$, etc.

Therefore $\bar{\alpha}_{p_0}0^+ \leqslant \bar{\alpha}_{p_1}1^+ \leqslant \bar{\alpha}_{p_2}2^+ \leqslant \ldots$.

Define a sequence α by taking $\bar{\alpha}r^+ =_{df} \bar{\alpha}_{p_r}r^+$ for each $r \in N$. This generates a sequence by the result of the previous paragraph and is well defined since $\bar{\alpha}r^+ = \bar{\alpha}s^+$ implies $lh\bar{\alpha}r^+ = lh\bar{\alpha}s^+$ i.e. $r = s$.

$p_r \leqslant p$ implies $\bar{\alpha}r^+ = \bar{\alpha}_{p_r}r^+ = \bar{\alpha}_p r^+$ so that

for each r, for some p_r, for each p, $p_r \leqslant p$ implies $\bar{\alpha}r^+ = \bar{\alpha}_p r^+$, where the least nni s with $\alpha s \neq \alpha_p s$ satisfies $r < s$ so that $d(\alpha_1\alpha_p) = \frac{1}{s^+} < \frac{1}{r^+}$. Therefore for each r, for some p_r, for each p, $p_r \leqslant p$ implies $d(\alpha,\alpha_p) < \frac{1}{r}$ i.e. $\alpha_0, \alpha_1, \ldots$ converge to α. Hence the Baire Space is complete.

<u>1.1.13. Definition</u>. A topological space X has the Baire Property $=_{df}$ each set of the 1st category (meagre) is a boundary set (i.e. its complement is dense).

<u>1.1.14. Remark</u>. This definition is equivalent to the more usual one: X has the Baire Property $=_{df}$ any intersection of a countable family of open dense subsets of X is dense.

For $\Rightarrow$)let $\cap_i A_i$ be any intersection of a countable family of open dense sets

implies $X-\cap_i A_i = \cup(X-A_i)$ and for each i

int $cl(X-A_i) = X - cl$ int $A_i = X - X = \phi$

implies $X-\cap_i A_i = \cup(X-A_i)$ and for each i, $X-A_i$ is nowhere dense

implies $X-\cap_i A_i = \cup(X-A_i)$ and $\cup(X-A_i)$ is meagre implies $X-\cap_i A_i$ is a boundary set by supposition

implies $\cap_i A_i$ is dense.

For $\Leftarrow$) let A be meagre, i.e. $A = \cup_i A_i$ for some countable family of nowhere dense sets

implies $X-\cup_i A_i = \cap(X-A_i)$ and for each i

$\quad$ $\text{cl int}(X-A_i) = X - \text{int cl } A_i = X$

implies $X-\cup_i A_i = \cap(X-A_i)$ and $\{\text{int}(X-A_i)\}$ is a countable set of open dense sets

implies $X-\cup_i A_i = \cap(X-A_i)$ is dense by supposition

implies $A = \cup_i A_i$ is a boundary set.

<u>1.1.15. Proposition</u>. Each complete metric space has the Baire Property.

<u>Proof</u>. Let X be a complete metric space and $A = \cup_i A_i$, where the A_i are nowhere dense, i.e. A is meagre. Recall that: a set is dense is equivalent to its intersection with each non-empty open set being non-empty. To show A is a boundary set we need show $-A$ is dense, i.e. for each non-empty open set B $-A \cap B \neq \phi$. This is equivalent to; for each non-empty open set B, not $B \subseteq A$.

Given a non-empty open set B we firstly prove by induction on r that there exist sets $\text{cl } B(x_r, \epsilon_r)$ (where $B(x_r, \epsilon_r)$ is the open ball centre x_r, radius ϵ_r) such that

$\text{cl } B(x_0, \epsilon_0) \subseteq B \cap -\text{cl } A_0$, $\text{cl } B(x_{r^+}, \epsilon_{r^+}) \subseteq B(x_r, \epsilon_r) \cap -\text{cl } A_{r^+}$ for each $r \in N$.

A_0 nowhere dense implies $\text{int cl } A_0 = \phi$

B open so $\text{int } B = B$ $\quad$ implies not $B \subseteq \text{cl } A_0$

$\quad$ implies there exists an x_0, $x_0 \in B \cap -\text{cl } A_0$

$B \cap -\text{cl } A_0$ open implies there exists $0 < \in_0 < 1$ such that

$\text{cl } B(x_0, \in_0) \subseteq B \cap -\text{cl } A_0$.

For each $r \in N$, given $B(x_r, \in_r)$, A_{r^+} nowhere dense

implies $\text{int cl } A_{r^+} = \phi$

implies not $B(x_r, \in_r) \subseteq \text{cl } A_{r^+}$

implies there exists an x_{r^+}, $x_{r^+} \in B(x_r, \in_r) \cap -\text{cl } A_{r^+}$

$B(x_r, \in_r) \cap -\text{cl } A_{r^+}$ open implies there is some

$0 < \in_{r^+} < \frac{1}{r^{++}}$ such that

$\text{cl } B(x_{r^+}, \in_{r^+}) \subseteq B(x_r, \in_r) \cap -\text{cl } A_{r^+}$.

For each $r_0 \in N$, each $r, s \geqslant r_0, x_r, x_s \in \text{cl } B(x_{r_0}, \in_{r_0})$, so that for each $r, s \geqslant r_0$, $d(x_r, x_s) \leqslant 2\in_0 < \frac{2}{r^+}$.

Therefore $d(x_r, x_s) \to 0$ as $r, s \to \infty$. Since X is a complete metric space there is some $x \in X$ with $d(x, x_r) \to 0$ as $r \to \infty$. Each $\text{cl } B(x_r, \in_r)$ is a closed set with $x_r, x_{r+1}, \ldots \in \text{cl } B(x_r, \in_r)$. Hence $x \in \text{cl } B(x_r, \in_r)$. Thus $x \in \text{cl } B(x_r, \in_r) \subseteq B$.

Suppose $x \in A$, then $x \in A_r$ for some r

implies $x \in \text{cl } A_r$

implies $x \notin B(x_{r^-}, \in_{r^-}) \cap -\text{cl } A_r$

implies $x \notin \text{cl } B(x_r, \in_r)$. Contradiction.

Therefore $x \notin A$ and hence not $B \subseteq A$.

<u>1.1.16. Proposition</u>. The Baire Space has the Baire Property.

<u>1.1.17. Remarks</u>. Even though the Baire Space is complete metric it is not compact since $\{< n > : n \in N\}$ is an open cover which has no finite subcover.

Because any number between 0 and 1 can be represented as a continuted fraction $a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \ldots$, it can be shown via the function which sends $a_1 + \lfloor\frac{1}{a_2}\rfloor + \lfloor\frac{1}{a_3}\rfloor + \ldots$ to the sequence $< a_1\ a_2\ a_3\ \ldots >$, that the set of irrationals between 0 and 1 (and so all irrationals) with the usual subspace topology is homeomorphic to the Baire Space.

1.2. CONTINUOUS FUNCTIONALS ON BAIRE SPACE

1.2.1. Remark. Continuous functionals of the type $N^N \to N$, $N^N \to N^N$ are to be used for the domain of the model of Intuitionistic second order arithmetic that we shall later construct. We now characterize such functionals in terms of functionals defined on the Baire Space neighbourhood basis, which will be seen to be an interpretation of Brouwer operations.

1.2.2. Definition. A function E from finite sequences into N is a neighbourhood function (nbdfn) $=_{df}$ for each sequence α, there are some x, y $\in$ N with

$$E(\bar{\alpha}z) = 0 \qquad \text{for } z < y \qquad E(\bar{\alpha}z) = x^+ \qquad \text{for } z \geq y.$$

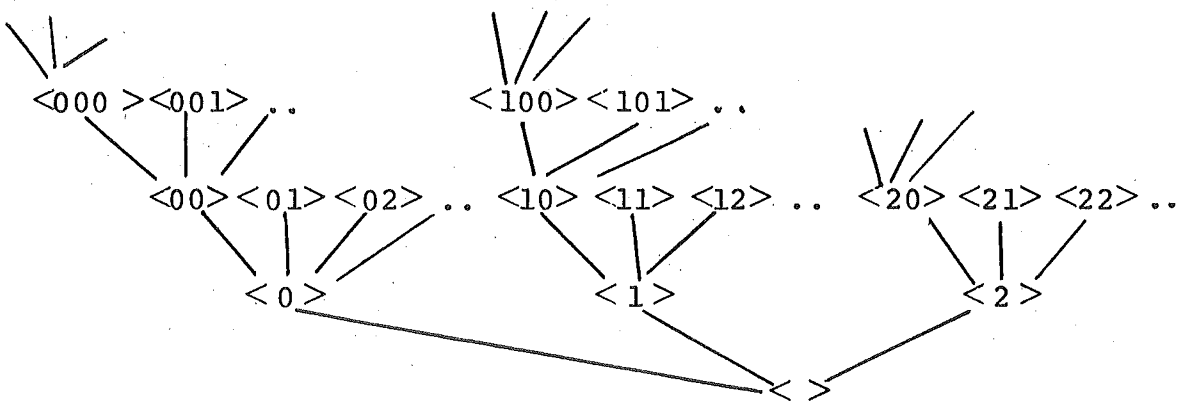
Each neighbourhood function E determines a functional Φ^E from N^N into N given by:

$$\Phi^E\alpha =_{df} x \qquad \text{iff for some } y, E(\bar{\alpha}y) = x^+.$$

Each neighbourhood function E determines a functional Ψ^E from N^N into N^N given by:

$$\Psi^E(\alpha)(y) =_{df} x \quad \text{iff for some } z,\ E(< y >{*}\ \bar{\alpha}z) = x^+.$$

1.2.3. Remark. As mentioned in 1.1.2 the set of finite sequences is a partially ordered set which has the following diagram



The 'branches' of the diagram are chains, i.e. for each m, n either $m \leqslant n$ or $n \leqslant m$. Hence given a branch B of this diagram, a nbdfn E maps members of B, whose length is less than some y^+, to 0, and maps members of B, whose length is greater than or equal to y^+, to x^+, for some x. Hence a nbdfn sends part of the branch to 0, the rest to a constant number. Therefore if $\alpha \in \{n\}$ and $En = x^+$ then $\Phi^E\alpha = x$, and if $\alpha \in \{<y>* n\}$ and $E(<y> * h)$ then $\Phi^E(\alpha)(y) = x$.

<u>1.2.4. Proposition.</u> If N^N has the Baire Space topology and N the discrete topology then a functional $\Phi : N^N \to N$ is continuous iff

$$(x)(\alpha)[\Phi\alpha = x \Rightarrow (Em)(\alpha \in \{m\} \;\&\; (\beta)(\beta \in \{m\} \Rightarrow \Phi\beta = x))],$$

where we use '(α)' for 'for each α', '$(E\beta)$' for 'there is some β', '$\Rightarrow$' for 'implies', '&' for 'and'.

<u>Proof.</u> Φ continuous implies $(x \in N)(\Phi^{-1}[\{x\}]$ is open)
implies $(x \in N)(\{\alpha : \Phi\alpha = x\}$ is a union of basis neighbourhoods)
implies $(x \in N)(\{\alpha : \Phi\alpha = x\}$ is covered by basis neighbourhoods which are contained by it)

implies $(x \in N)(\{\alpha : \Phi\alpha = x\}$ is covered by basis neighbourhoods which are contained by it)

implies $(x \in N)(\alpha)[\Phi\alpha = x \Rightarrow (Em)(\alpha \in \{m\} \& \{m\} \subseteq \{\alpha : \Phi\alpha = x\})]$

implies $(x)(\alpha)[\Phi\alpha = x \Rightarrow (Em)(\alpha \in \{m\} \& (\beta)(\beta \in \{m\} \Rightarrow \Phi\beta = x))]$.

$(x)(\alpha)[\Phi\alpha = x \Rightarrow (Em)(\alpha \in \{m\} \ \& \ (\beta)(\beta \in \{m\} \Rightarrow \Phi\beta = x))]$

implies $(A)(\alpha)[\Phi\alpha \in A \Rightarrow (Em)(\alpha \in \{m\} \& (\beta)(\beta \in \{m\} \Rightarrow \Phi\beta \in A))]$

[since $\Phi\alpha \in A \Rightarrow (Ex \in A)(\Phi\alpha = x)$ and so $(Ex \in A)(\Phi\beta = x)$

implies $(A)(\alpha)[\Phi\alpha \in A \Rightarrow (Em)(\alpha \in \{m\} \& \{m\} \subseteq \{\alpha : \Phi\alpha \in A\})]$

implies $(A \subseteq N)(\{\alpha : \Phi\alpha \in A\}$ is covered by basis neighbourhoods which are contained by it)

implies $(A \subseteq N)(\{\alpha : \Phi\alpha \in A\}$ is a union of basis neighbourhoods)

implies $(A \subseteq N)(\Phi^{-1}[A]$ is open) i.e. Φ is continuous.

1.2.5. Proposition. For each nbdfn E, Φ^E is continuous.

Proof. Given $x \in N$, $\alpha \in N^N$ such that $\Phi^E\alpha = x$, by definition, for some y, $E(\bar{\alpha}y) = x^+$. Considering this $\bar{\alpha}y$, we have $\alpha \in \{\bar{\alpha}y\}$ and for each β, $\beta \in \{\bar{\alpha}y\}$ implies $\bar{\beta}y = \bar{\alpha}y$

implies $E(\bar{\beta}y) = E(\bar{\alpha}y) = x^+$

implies $\Phi^E\beta = x$.

1.2.6. Proposition. For each continuous functional $\Phi : N^N \to N$, there is a nbdfn E with $\Phi = \Phi^E$.

Proof. For each finite sequence m define a functional Φ_m from N^N into N by $\Phi_m\alpha =_{df} \Phi(m * \alpha)$ for each $\alpha \in N^N$.

Lemma. If for each $x \in N$ there is a nbdfn $E^{m*\langle x\rangle}$ with $\Phi_{m*\langle x\rangle} = \Phi^{E^{m*\langle x\rangle}}$ then there is a nbdfn E^m with $\Phi_m = \Phi^{E^m}$.

Proof. Define E^m by $E^m <> =_{df} 0$, $E^m(<x> * n) =_{df} E^{m*<x>}n$ for each x, n.

For each α, $E^m(\bar{\alpha}z) = E^m(<\alpha 0> * \bar{\beta}z^-)$ where $\beta r = \alpha r^+$ for each r

$$= E^{m*<\alpha 0>}(\bar{\beta}z^-) = 0 \quad \text{iff } z^- < y_{\alpha 0}$$

$$= x^+_{\alpha 0} \quad \text{iff } z^- \geqslant y_{\alpha 0}$$

[for some $x_{\alpha 0}$, $y_{\alpha 0}$ since $E^{m*<\alpha 0>}$ is a nbdfn].

Hence there is some $x_{\alpha 0}$, $y^+_{\alpha 0}$ such that $E^m(\bar{\alpha}z) = 0$ iff $z < y^+_{\alpha 0}$, $E^m(\bar{\alpha}z) = x^+_{\alpha 0}$ iff $z \geqslant y^+_{\alpha 0}$, i.e. E^m is a nbdfn.

For each sequence $<x> * \alpha$,

$$\begin{aligned}
\Phi_m(<x> * \alpha) &= \Phi(m*<x> * \alpha) && \text{by definition}\\
&= \Phi_{m*<x>}\alpha && \text{by definition}\\
&= \Phi^{E^{m*<x>}}\alpha && \text{by supposition}\\
&= (E^{m*<x>}(\bar{\alpha}y))^- && \text{for some y, by definition}\\
&= (E^m(<x> * \bar{\alpha}y))^- && \text{for some y, by definition}\\
&= (E^m(\overline{<x> * \alpha}y^+))^- && \text{for some } y^+\\
&= \Phi^{E^m}(<x> * \alpha) && \text{by definition.}
\end{aligned}$$

Hence $\Phi^{E^m} = \Phi_m$.

Now assuming that there is no nbdfn $E^{<>}$ such that $\Phi_{<>} = \Phi = \Phi^{E^{<>}}$, a contradiction can be obtained. By induction we define a sequence α such that for each $r \in N$ there is no nbdfn $E^{\bar{\alpha}r}$ with $\Phi_{\bar{\alpha}r} = \Phi^{E^{\bar{\alpha}r}}$.

For $r = 0$, our supposition is that there is no nbdfn $E^{<>}$ such that $\Phi_{<>} = \Phi^{E^{<>}}$.

For each r, assume there is some finite sequence m with lhm = r such that there is no nbdfn E^m with $\Phi_m = \Phi^{E^m}$. By the lemma there is some $x_r \in N$ such that there is no nbdfn $E^{m*<x_r>}$ with $\Phi_{m*<x_r>} = \Phi^{E^{m*<x_r>}}$.

Define α by taking $\alpha(r) =_{df} x_r$ for each r.

Φ is continuous implies there is some finite sequence m with $\alpha \in \{m\}$ such that $\Phi\beta = \Phi\alpha$ for each $\beta \in \{m\}$.

For each $\gamma \in N^N$, $\Phi_{\bar{\alpha}lhm}\gamma = \Phi_m\gamma$ since $\alpha \in \{m\}$ implies $\bar{\alpha}lhm = m$

$= \Phi(m*\gamma) = \Phi\alpha$ since $m*\gamma \in \{m\}$ and Φ is continuous.

Thus $\Phi_{\bar{\alpha}lhm}$ is a constant functional with

$$\Phi_{\bar{\alpha}lhm} = \Phi^{E^{\bar{\alpha}lhm}} \quad \text{where } E^{\bar{\alpha}lhm}\, n = (\Phi\alpha)^+ \text{ for each n.}$$

This contradicts the property which α had by construction, that for each $r \in N$ there is no nbdfn $E^{\bar{\alpha}r}$ with $\Phi_{\bar{\alpha}r} = \Phi^{E^{\bar{\alpha}r}}$.

Therefore there is a nbdfn $E^{<>} = E$ such that $\Phi = \Phi_{<>} = \Phi^{E^{<>}}$.

<u>1.2.7. Remarks</u>. The proof of 1.2.6 is due to G. Kreisel and A.S. Troelstra and appears in [11].

Combining 1.2.5, 1.2.6 we have each continuous functional $\Phi : N^N \to N$ is continuous iff $\Phi = \Phi^E$ for some nbdfn E. It is important to note that while 1.2.5 is intuitionistically provable, 1.2.6, since it uses the principle of excluded middle, is extremely classical and there is no intuitionistic proof of it. Nevertheless, as we shall see, the Intuitionists hold that this equivalence (and even something

stronger: that each functional is continuous) is true and justify it on other grounds.

We have the analogues of 1.2.5, 1.2.6 for the higher order functionals from N^N into N^N. We now prove these noting that we can easily prove as in the manner of 1.2.4 that $\Psi : N^N \to N^N$ is continuous iff

$(n)(\alpha)[\Psi\alpha \in \{n\} \Rightarrow (Em)(\alpha \in \{m\} \;\&\; (\beta)(\beta \in \{m\} \Rightarrow \Psi\beta \in \{n\}))]$

1.2.8. Proposition. For each nbdfn E, Ψ^E is continuous.

Proof. Given finite sequence n, $\alpha \in N^N$,

$$\begin{aligned} \Psi^E\alpha \in \{n\} &\text{ implies } (\Psi^E\alpha)(i) = ni \quad \text{for each } i \leqslant lhn \\ &\text{ implies } E(i*\bar{\alpha}z_i) = (ni)^+ \text{ for some } z_i, \text{ for each } i \leqslant lhn . \end{aligned}$$

Let $z = \max_{i \leqslant lhn} z_i$ and $m = \bar{\alpha}z$. Hence $\alpha \in \{m\}$, and

$$\begin{aligned} \beta \in \{m\} &\text{ implies } \bar{\beta}z_i = \bar{\alpha}z_i \quad \text{for each } i \leqslant lhn \\ &\text{ implies } i*\bar{\beta}z_i = i*\bar{\alpha}z_i \quad \text{for each } i \leqslant lhn \\ &\text{ implies } E(i*\bar{\beta}z_i) = E(i*\bar{\alpha}z_i) = (ni)^+ \text{ for each } i \leqslant lhn \\ &\text{ implies } \Psi^E(\beta)(i) = ni \quad \text{for each } i \leqslant lhn \\ &\text{ implies } \Psi^E(\beta) \in \{n\}, \quad \text{i.e. } \Psi^E \text{ is continuous.} \end{aligned}$$

1.2.9. Proposition. For each continuous functional $\Psi : N^N \to N^N$, there is a nbdfn E such that $\Psi = \Psi^E$.

Proof. For each finite sequence n define a functional Ψ_n by

$\Psi_n(\alpha)(y) =_{df} \Psi(tln*\langle y\rangle*\alpha)(n0)$ where $n = \langle n0\rangle*tln$, for each α, y.

Lemma. If for each $x \in N$ there is a nbdfn $E^{n*\langle x\rangle}$ such that $\Psi_{n*\langle x\rangle} = \Psi^{E^{n*\langle x\rangle}}$, then there is a nbdfn E^n such that $\Psi_n = \Psi^{E^n}$.

<u>Proof</u>. Define E^n by $E^n\langle\,\rangle =_{df} 0$, $E^n(\langle x\rangle * m) =_{df} E^{n*\langle x\rangle}m$

for each x, m.

For each α, $E^n(\bar{\alpha}z) = E^n(\langle\alpha 0\rangle * \bar{\beta}z^-)$ where $\beta r = \alpha r^+$ for each r

$= E^{n*\langle\alpha 0\rangle}(\bar{\beta}z^-) = 0$ iff $z < y_{\alpha 0}$

$= x^+_{\alpha 0}$ iff $z \geqslant y_{\alpha 0}$

[for some $x_{\alpha 0}$, $y_{\alpha 0}$ since $E^{n*\langle\alpha 0\rangle}$ is a nbdfn].

Therefore for some $x_{\alpha 0}$, $y^+_{\alpha 0}$, $E^n(\bar{\alpha}z) = 0$ iff $z < y^+_{\alpha 0}$, $E^n(\bar{\alpha}z) = x^+_{\alpha 0}$ iff $z \geqslant y^+_{\alpha 0}$. Hence E^n is a nbdfn.

For each α, y, $\Psi^{E^n}(\alpha)(y) = (E^n(\langle y\rangle * \bar{\alpha}z))^-$ for some z

by definition

$= (E^{n*\langle y\rangle}(\bar{\alpha}z))^-$ for some z by definition

$= (E^{n*\langle y\rangle}(\langle\alpha 0\rangle * \overline{tl(\alpha)}z^-))^-$ for some z

[where $\overline{tl(\alpha)}(z) = tl(\bar{\alpha}z^+)$]

$= \Psi^{E^{n*\langle y\rangle}}(tl(\alpha))(\alpha 0)$ by definition

$= \Psi_{n*\langle y\rangle}(tl(\alpha))(\alpha 0)$ by supposition

$= \Psi(tl((n*\langle y\rangle) * \langle\alpha 0\rangle * tl(\alpha))((n*\langle y\rangle)0)$ by definition

$= \Psi(tln * \langle y\rangle * \alpha)(n0)$

[since $tl(n*\langle y\rangle) = tl(n)*\langle y\rangle$, $\alpha = \langle\alpha 0\rangle * tl(\alpha)$,

$(n*\langle y\rangle)0 = n0$]

$= \Psi_n(\alpha)(y)$.

Hence $\Psi_n = \Psi^{E^n}$.

Assuming there is no nbdfn $E = E^{\langle\,\rangle}$ such that $\Psi = \Psi_{\langle\,\rangle} = \Psi^{E^{\langle\,\rangle}}$ a contradiction is obtained. By induction we define a sequence α such that, for each $r \in N$, there is no nbdfn $E^{\bar{\alpha}r}$ with $\Psi_{\bar{\alpha}r} = \Psi^{E^{\bar{\alpha}r}}$.

For $r = 0$, our supposition is that there is no nbdfn $E^{<>}$ such that $\Psi_{<>} = \Psi^{E^{<>}}$.

For each r, we assume there is some finite sequence n with lhn = r such that there is no nbdfn E^n with $\Psi_n = \Psi^{E^n}$. By the lemma there is some x_r such that there is no nbdfn $E^{n*<x_r>}$ with $\Psi_{n*<x_r>} = \Psi^{E^{n*<x_r>}}$.

Define α by taking $\alpha(r) =_{df} x_r$ for each r.

<u>Lemma</u>. The functional $\Phi : N^N \to N$ with $\Phi(\gamma) = \Psi(\gamma)(\alpha 0)$ is continuous.

<u>Proof</u>. Given γ, x, $\Phi(\gamma) = x$ implies $\Psi(\gamma)(\alpha 0) = x$

implies there is some $n = \overline{\Psi\gamma}(\alpha 0)^+$ such that

$\Psi\gamma \in \{n\}$.

Since Ψ is continuous, there is some m such that $\gamma \in \{m\}$ and $\beta \in \{m\}$ implies $\Psi\beta \in \{n\}$.

With this $\{m\}$, $\beta \in \{m\}$ implies $\Psi\beta \in \{n\}$

implies $\overline{\Psi\beta}(\alpha 0)^+ = \overline{\Psi\gamma}(\alpha 0)^+$

implies $\Psi\beta(\alpha 0) = x$ i.e. $\Phi(\beta) = x$.

Thus by 1.2.6 $\Phi = \Phi^E$ for some nbdfn E.

Defining β by $\beta r = \alpha r^+$ for each r, since $\Phi(\beta) = w$ for some w, using $\Phi = \Phi^E$ we have $E(\bar{\beta} z) = w^+$ for some z.

With this z, for each γ and y,

$\Psi_{\bar{\alpha} z^+}(\gamma)(y) = \Psi(tl(\bar{\alpha} z^+)*<y>*\gamma)(\alpha 0)$ by definition

$= \Phi(tl(\bar{\alpha} z^+)*<y>*\gamma)$ by definition

$= \Phi^E(\bar{\beta} z * <y> * \gamma)$

[since $\beta r = \alpha r^+$ for each r and $\Phi = \Phi^E$ for some E]

$= w$

[since Φ^E is continuous and $\Phi^E(\beta) = w$ and

$\bar{\beta}z * \langle y \rangle * \gamma \in \{\bar{\beta}z\}$].

Therefore $\Psi_{\bar{\alpha}z^+}$ is constant. Moreover if $E_{\bar{\alpha}z^+}$ is defined by $E^{\bar{\alpha}z^+}(n) = E(\bar{\beta}z)$ for each n, then $\Psi_{\bar{\alpha}z} = \Psi^{E^{\bar{\alpha}z^+}}$. This contradicts the property $\Psi_{\bar{\alpha}r}$ had by construction, that for each r, there is no nbdfn $E^{\bar{\alpha}r}$ with $\Psi_{\bar{\alpha}r} = \Psi^{E^{\bar{\alpha}r}}$.

Therefore there is a nbdfn $E^{<>} = E$ such that $\Psi = \Psi_{<>} = \Psi^{E^{<>}}$.

<u>1.2.10. Remarks</u>. This classical proof of 1.2.9 which is due to R.A. Bull, does not appear in the literature, nor is it hinted at in [11]. Yet it is only a modification of 1.2.6, using 1.2.6.

Furthermore what 1.2.9 states is provable for the Intuitionists under the same assumptions that makes 1.2.6 provable for them. So we would expect G. Kreisel and A.S. Troelstra to use 1.2.9 to give a classical interpretation to Ψ^E.

1.3. A PRINCIPLE OF INDUCTION FOR FINITE SEQUENCES.

<u>1.3.1. Remarks</u>. In this section we will derive a principle of induction for finite sequences which enables us to redefine nbdfns without referring to infinite sequences.

Ordinary induction for integers has the following form: if Ax is any statement about x, for each $x \in N$, then

$$(A0 \ \& \ (x)(Ax \Rightarrow Ax^+)) \Rightarrow (x)Ax.$$

We can derive from this an alternative version:

$$(x)((y)(x > y \Rightarrow Ay) \Rightarrow Ax) \Rightarrow (x)Ax.$$

In this form we see that given x, if everything 'related' to x (in this case $>$ is the relation) satisfying A implies x satisfies A, then A is satisfied by all x. It is not at all obvious, but when we use the relation $x = y^+$ in place of $x > y$ in the above schema, we have something equivalent to the usual formulation of induction.

From considerations such as these we might generalize to allow any relation at all, to give a principle of induction on R called transfinite induction TI(R) :

$$(x)((y)(xRy \Rightarrow Ay) \Rightarrow Ax) \Rightarrow (x)Ax \; .$$

However this may not hold for just any R. A condition on R for which TI(R) holds is that of well-foundedness (WF(R)): $(\alpha)(Ex)\sim(\alpha x)R(\alpha x^+)$. This condition prevents infinite R chains e.g. $(\alpha_0)R(\alpha_1)$, $(\alpha_1)R(\alpha_2)$,... Indeed both relations $x > y$, $x = y^+$ can be easily seen to be well-founded.

Thus we have a general principle of induction $WF(R) \Rightarrow TI(R)$, which in particular implies the ordinary principle of induction. We shall use this general principle of induction to derive a principle of induction for finite sequences.

<u>1.3.2. Remarks.</u> In the proof of 1.1.5 we assigned, 1-1 and onto, nnis to finite sequences. So at the one time we can regard an nni as both an nni and a sequence which is assigned that nni. This 'blurring' technique is widely used in the following work.

Suppose that Ax is a statement about x, for each $x \in N$, such that $(\alpha)(Ey)A(\bar{\alpha}y)$ and $(m)(n)[Am \Rightarrow A(m*n)]$ so that

$$(\alpha)(Ey)(z)[\, y \leqslant z \Rightarrow A(\bar{\alpha}z)] \qquad (*).$$

Then A determines a relation on finite sequences with

$$mRn =_{df} (Ex)(m * \langle x\rangle = n) \;\&\; \sim An.$$

If β is a sequence such that $(Ex)(\bar{\beta}y * \langle x\rangle = \bar{\beta}y^{+}) \;\&\sim A(\bar{\beta}y^{+})$, $(Ex)(\bar{\beta}y^{+} * \langle x\rangle = \bar{\beta}y^{++}) \;\&\sim A(\bar{\beta}y^{++}), \ldots,$ for some y, then β would not satisfy condition ⊛ on A. Hence the relation R satisfies the 'well foundedness' condition $(\beta)(y)(Ez)(\sim \bar{\beta}(y+z)R\bar{\beta}(y+z)^{+})$.

<u>1.3.3. Proposition</u>. For the relation R defined above, WF(R).

<u>Proof</u>. Given α, the values αx are nnis assigned to certain finite sequences. If there is some r with not $(Ex)(\alpha r * \langle x\rangle = \alpha r^{+})$ then $(Ex) \sim (\alpha x)\, R\, (\alpha x^{+})$.

If for each $r \in N$, $(Ex)(\alpha r * \langle x\rangle = \alpha r^{+})$ then $\alpha 1$ is not the empty sequence and $\alpha r^{+} = \alpha 1 * \langle x_1\rangle * \ldots * \langle x_r\rangle$, for some $x_1, \ldots, x_r \in N$, for each $r \in N$. In this case define a sequence β by, $\beta 0 =_{df} (\alpha 1)0, \ldots, \beta(lh\alpha 1)^{-} =_{df} (\alpha 1)(lh\alpha 1)^{-}$, $\beta(r + (lh\alpha 1)^{-}) =_{df} (\alpha r^{+})(r + (lh\alpha 1)^{-})$ for each $r \in N$. With this, $\alpha 1 = \bar{\beta}(lh\alpha 1)$, $\alpha r^{+} = \bar{\beta}((lh\alpha 1)+r)$ for each $r \in N$. So applying the 'well-foundedness' condition of 1.3.2 with lh $\alpha 1$ for y, x^{+} for z, $\sim(\alpha x)R(\alpha x^{+})$.

<u>1.3.4. Remarks</u>. Showing that R satisfies the 'well-foundedness' condition and that WF(R) is due to A.Q. Abraham from an idea of R.A. Bull.

The proof of 1.3.5 from WF(R) $\Rightarrow$ TI(R) of W.A. Howard and G. Kreisel in [7] appears unintelligible and the proof of this given by Van Dalen in [20] is incorrect.

1.3.5. Proposition. $[(\alpha)(Ey)A(\bar{\alpha}y) \& (m)(n)(Am \Rightarrow (m*n))$

$$\& (m)(Am \Rightarrow Bm) \& (m)((x)B(m*\langle x\rangle) \Rightarrow Bm)] \Rightarrow (m)Bm.$$

Proof. We have seen in 1.3.2, 1.3.3 that given $(\alpha)(Ey)A(\bar{\alpha}y)$ and $(m)(n)(Am \Rightarrow A(m*n))$ we can define a relation R by $mRn =_{df} (Ex)(m * \langle x\rangle = n) \& \sim An$ such that WF(R).

TI(R) is $(m)((n)(((Ex)(m*\langle x\rangle = n) \& \sim An) \Rightarrow Bn) \Rightarrow Bm) \Rightarrow (m)Bm$.

Supposing $(m)(Am \Rightarrow Bm)$ and $(m)((x)B(m * \langle x\rangle) \Rightarrow Bm)$, for given m $(n)(((Ex)(m * \langle x\rangle = n) \& \sim An) \Rightarrow Bn)$

implies $(n)(((Ex)(m*\langle x\rangle = n) \& \sim An) \Rightarrow Bn)$

and $(n)(((Ex)(m*\langle x\rangle = n) \& An) \Rightarrow Bn)$

[since $(m)(Am \Rightarrow Bm)$ and $(p \Rightarrow r) \Rightarrow ((q \& p) \Rightarrow r)$.]

implies $(n)((Ex)(m * \langle x\rangle = n) \Rightarrow Bn)$

[since $(((p \& \sim q) \Rightarrow r) \& ((p \& q) \Rightarrow r)) \Rightarrow (p \Rightarrow r)$]

implies $(n)(x)(m * \langle x\rangle = n \Rightarrow Bn)$

[since $(Ex)Ax \Rightarrow P$ iff $(x)(Ax \Rightarrow P)$]

implies Bm

[since $(m)((x)B(m * \langle x\rangle) \Rightarrow Bm)$ and $m * \langle x\rangle = m * \langle x\rangle$].

Therefore the induction step for TI(R) holds, so that (m)Bm.

1.3.6. Remarks. This principle of induction for finite sequences is known as bar induction (BI) and is due to Brouwer. BI, here, has the conclusion (m)Bm. This can be weakened to $B\langle\rangle$ and yet remain as strong a principle. For $(m)Bm \Rightarrow B\langle\rangle$ is trivial. To derive the general conclusion (m)Bm from the specific $B\langle\rangle$, note by recursion theory it can be shown that if $g = f_n =_{df} (x)((x < lhn \Rightarrow g(x) = nx)$ $\& (x \geqslant lhn \Rightarrow g(x) = f(x - lhn)))$ then $(n)(f)(Eg)(g = f_n)$ - (1).

Then by (1) and $(\alpha)(Ey)A\bar{\alpha}y$ we can show that for each c, $(\alpha)(Ey)A(c * \bar{\alpha}y)$. Trivially we can show

$(m)(n)(A(c*m) \Rightarrow A(c*m*n))$, $(m)(A(c*m) \Rightarrow B(c*m))$, $(m)((x)B(c*m*\langle x\rangle) \Rightarrow B(c*m))$ from the other hypotheses of BI. Hence, by BI we can conclude for each c, $B(c*\langle\,\rangle)$, i.e. $(c)Bc$ or by change of variables $(m)Bm$.

We now show that from BI we can conclude $WI(R) \Rightarrow TI(R)$, for any relation R, and hence for our particular relation, so that BI and $WI(R) \Rightarrow TI(R)$ are equivalent. The proof is due to W.A. Howard and G. Kreisel in [7].

1.3.7. Proposition. BI implies $WF(R) \Rightarrow TI(R)$ for any R.

Proof. Let R be any relation, assume $WF(R)$

i.e. $(\alpha)(Ex) \sim (\alpha x)R(\alpha x^{+})$.

Then we have $(\alpha)(Ex) \sim (y \leq x)(\alpha y)R(\alpha y^{+})$ ——— 1.

Suppose $(x)((y)(xRy \Rightarrow Ay) \Rightarrow Ax)$ ——— 2.

i.e. the induction step for $TI(R)$.

Define $Bn =_{df} (y < lhn \dot{-} 1)(nyRny^{+})$ for $lhn \geq 2$ and Bn is true for $lhn < 2$,

$$Pn =_{df} \sim Bn,\quad Qn =_{df} (lhn = 0\ \&\ (y)Ay)$$

$$\text{or } (lhn \neq 0\ \&\ [Bn \Rightarrow (y)(n(lhn \dot{-} 1)Ry \Rightarrow Ay)]).$$

$$(\alpha)(Ex)P\bar{\alpha}x =_{df} (\alpha)(Ex) \sim (y < lh\bar{\alpha}x \dot{-} 1)(\bar{\alpha}x)yR(\bar{\alpha}x)y^{+}$$

$$= (\alpha)(Ex) \sim (y < x)(\alpha yR\ \alpha y^{+}) \quad \text{which holds by } 1$$

$$[\text{since } lh\bar{\alpha}x = x^{+} \text{ and for } y < lh\bar{\ }x \dot{-} 1,\ (\bar{\alpha}y)y = \alpha y,\ (\bar{\alpha}x)y^{+} = \alpha y].$$

Given m, n, Pm implies $\sim (y < lhm \dot{-} 1)(myRmy^{+})$

implies $(Ey < lhm \dot{-} 1) \sim (myRmy^{+})$

implies $(Ey < lhm*n \dot{-} 1) \sim (myRmy^{+})$

[since $lhm*n > lhm$ and for $y < lhm \dot{-} 1$, $(m*n)y = my$]

implies $\sim (y < lhm*n \dot{-} 1)(myRmy^{+})$

implies $P(m*n)$, i.e. $(m)(n)(Pm \Rightarrow P(m*n))$.

Given m, Pm implies Pm and either lhm = 0 or lhm $\neq$ 0;

Pm and lhm = 0 implies Pm and Bm

[since Bm is true for lhm < 2]

implies Qm

[since Bm $=_{df}$ $\sim$ Pm and $(p \Rightarrow (\sim p \Rightarrow q))$];

Pm and lhm $\neq$ 0 implies lhm $\neq$ 0 and $\sim$ Bm

implies lhm $\neq$ 0 and $Bm \Rightarrow ((y)(m(lhm \dot{-} 1)Ry \Rightarrow Ay))$

[since $(p \Rightarrow (\sim p \Rightarrow q))$ and using modus ponens]

implies $(lhm = 0 \ \& \ (y)Ay)$

or $(lhm \neq 0 \ \& \ [Bm \Rightarrow ((y)(m(lhm \dot{-} 1)Ry \Rightarrow Ay)])$

[since $p \Rightarrow (p \vee q)$]

implies Qm by definition;

i.e. $(m)(Pm \Rightarrow Qm)$.

Given m, $(x)Q(m * \langle x \rangle)$ implies $(x)Q(m * \langle x \rangle)$ and either lhm = 0 or lhm $\neq$ 0. $(x)Q(m * \langle x \rangle)$ and lhm = 0

implies lhm = 0 and $(x)Q\langle x \rangle$

implies lhm = 0 and

$(x)(B\langle x \rangle \Rightarrow (y)(\langle x \rangle \ lh\langle x \rangle \dot{-} 1 \ Ry \Rightarrow Qy))$

[since $lh\langle x \rangle \neq 0$]

implies lhm = 0 and $(x)B\langle x \rangle \Rightarrow (x)(y)(xRy \Rightarrow Ay)$

[since $(x)(Ax \Rightarrow Bx) \Rightarrow ((x)Ax \Rightarrow (x)Bx)$

and $lh\langle x \rangle \dot{-} 1 = 0$]

implies lhm = 0 and $(x)B\langle x \rangle \Rightarrow (x)Ax$

[by 2]

implies lhm = 0 and $(x)Ax$

[since for each x, $lh\langle x \rangle < 2$ and so

$B\langle x \rangle$ is true]

implies $(lhm = 0 \ \& \ (x)Ax)$ or

$(lhm \neq 0 \ \& \ [Bm \Rightarrow (y)(m(lhm \dot{-} 1)Ry \Rightarrow Ay)])$.

$(x)Q(m * \langle x\rangle)$ and $lhm \neq 0$

implies $lhm \neq 0$ and $(x)(B(m*\langle x\rangle) \Rightarrow (y)(m*\langle x\rangle(lhm*\langle x\rangle \dot{-} 1)Ry \Rightarrow Ay))$

[since $lhm * \langle x\rangle \neq 0$]

implies $lhm \neq 0$ and $(x)(B(m * \langle x\rangle) \Rightarrow (y)(xRy \Rightarrow Ay))$

[since $m * \langle x\rangle(lhm * \langle x\rangle \dot{-} 1) = x$]

implies $lhm \neq 0$ and $(x)(B(m * \langle x\rangle) \Rightarrow Ax)$

[by 2]

implies $lhm \neq 0$ and

$(x)((y < lhm \dot{-} 1)((m*\langle x\rangle)yR(m*\langle x\rangle)y^{+} \;\&\; mlhm \dot{-} 1Rx \Rightarrow Ax)$

implies $lhm \neq 0$ and

$(x)((y < lhm \dot{-} 1)(myRmy^{+} \;\&\; mlhm \dot{-} 1Rx) \Rightarrow Ax)$

implies $lhm \neq 0$ and

$(x)((y < lhm \dot{-} 1)myRmy^{+} \Rightarrow (mlhm \dot{-} 1\ Rx \Rightarrow Ax))$

implies $lhm \neq 0$ and

$(y < lhm \dot{-} 1)myRmy^{+} \Rightarrow (x)(mlhm \dot{-} 1\ Rx \Rightarrow Ax))$

implies $lhm \neq 0$ and $Bm \Rightarrow (x)(mlhm \dot{-} 1\ Rx \Rightarrow Ax)$

implies $lhm \neq 0$ and $Bm \Rightarrow (y)(mlhm \dot{-} 1\ Ry \Rightarrow Ay)$

implies $(lhm = 0 \;\&\; (y)Ay)$ or

$(lhm \neq 0 \;\&\; [Bm \Rightarrow (y)(mlhm \dot{-} 1\ Ry \Rightarrow Ay)])$

implies Qm. I.e. $(m)((x)Q(m * \langle x\rangle) \Rightarrow Qm)$.

Thus the four hypotheses of BI are satisfied so we can conclude (either indirectly or directly by 1.3.6) $Q\langle\rangle$, i.e. $(y)Ay$ or $(x)Ax$.

<u>1.3.8. Remarks</u>. Recall that a nbdfn E was defined as

$(\alpha)(Ex)(Ey)(z)((z < y \Rightarrow E\bar{\alpha}z = 0) \;\&\; (z \geqslant y \Rightarrow E\bar{\alpha}z = x^{+}))$.

When we replace $\bar{\alpha}z$ by the nni assigned to it, E becomes a sequence which we denote here by ε.

Let Nfn be the class of sequences ε such that

$(\alpha)(Ex)(Ey)(\varepsilon(\bar{\alpha}y) = x^+)$ ① and

$(m)(n)(x)[\varepsilon m = x^+ \Rightarrow \varepsilon(m * n) = x^+]$ ②.

Obviously Nfn contains all sequences which we associated with nbdfns. The converse is true as the following argument shows: Suppose given α, for some x, y, $\varepsilon(\bar{\alpha}y) = x^+$. Then there is a least y, say y', such that $\varepsilon(\bar{\alpha}y) = x^+$. Suppose $z < y'$ and $\varepsilon(\bar{\alpha}z) \neq 0$. Then if $\varepsilon(\bar{\alpha}z) = w^+$ then by ②
$x^+ = \varepsilon(\bar{\alpha}y') = \varepsilon(\bar{\alpha}z * m) = \varepsilon(\bar{\alpha}z) = w^+$, for suitable m with $y' = z * m$. Hence $w^+ = x^+$ and so y' is not the least y such that $\varepsilon(\bar{\alpha}y) = x^+$. Contradiction.

It follows from ① and ② that $(\alpha)(Ey)(\varepsilon(\bar{\alpha}y) \neq 0)$ and that $(m)(n)[\varepsilon m \neq 0 \Rightarrow \varepsilon(m * n) \neq 0]$. Thus the first two hypotheses of BI of the statement $\varepsilon n \neq 0$ are satisfied.

Define, analogously to Φ_m in 1.2.6, α_m as follows: $\alpha_m n =_{df} \alpha(m * n)$. Using these considerations we have the following.

<u>1.3.9. Proposition.</u> BI implies Nfn is the least class P of sequences such that $(Ex)(y)(\varepsilon y = x^+) \Rightarrow \varepsilon \in P$ ① and $(\varepsilon\langle\rangle = 0 \ \& \ (x)(\varepsilon_{\langle x\rangle} \in P)) \Rightarrow \varepsilon \in P$ ②.

<u>Proof.</u> If ε is a sequence such that $(Ex)(y)(\varepsilon y = x^+)$ then obviously $\varepsilon \in$ Nfn. If ε is a sequence with $\varepsilon_{\langle\rangle} = 0$ and $(x)(\varepsilon_{\langle x\rangle} \in \text{Nfn})$ then by the following argument (similar to that for E^m, $E^{m*\langle x\rangle}$ in 1.2.6), $\varepsilon \in$ Nfn.

For each sequence α, $\varepsilon(\bar{\alpha}z) = \varepsilon(\langle\alpha 0\rangle * \bar{\beta}z^-)$ where $\beta r = \alpha r^+$ for each r

$$\begin{aligned} &= \varepsilon_{\langle\alpha 0\rangle}(\bar{\beta}z^-) = 0 \quad \text{iff } z < y_{\alpha 0} \\ &= x^+_{\alpha 0} \quad \text{iff } z \geqslant y_{\alpha 0} \end{aligned}$$

for some $x_{\alpha 0}$, $y_{\alpha 0}$ since $(x)(\varepsilon_{\langle x\rangle} \in \text{Nfn})$.

It follows that $\varepsilon m = x^+ \Rightarrow \varepsilon(m * n) = x^+$, taking any sequence α with $\alpha \in \{m * n\}$. Thus $\varepsilon \in Nfn$ so that Nfn is a class of sequences that satisfies ① and ②.

Given a class P of sequences which satisfies these conditions, we use BI on the statements $\varepsilon x \neq 0$, $\varepsilon_x \in P$ about x, to show $Nfn \subseteq P$. We have seen in 1.3.8 that if $\varepsilon \in Nfn$ then the statement $\varepsilon x \neq 0$ satisfies the first two hypotheses of BI.

For a third hypothesis for BI, for each m,

$\varepsilon m \neq 0$ implies for each n, $\varepsilon(m * n) = \varepsilon m \neq 0$ since $\varepsilon \in Nfn$

implies for each n, $\varepsilon_m n = \varepsilon m \neq 0$

implies for some x, for each n, $\varepsilon_m n = x^+$

take x to be εm^-

implies $\varepsilon_m \in P$ by ①.

For the fourth hypothesis for BI, for each m, either $\varepsilon m = 0$ or $\varepsilon m \neq 0$.

$\varepsilon m \neq 0$ implies $\varepsilon_m \in P$ as above

implies $(x)(\varepsilon_{m*<x>} \in P) \Rightarrow \varepsilon_m \in P$

[by $p \Rightarrow (q \Rightarrow p)$ and modus ponens].

$\varepsilon m = 0$ implies if $(x)(\varepsilon_{m*<x>} \in P)$ then $\varepsilon_m<> = 0$)

and $(x)(\varepsilon_{m*<x>} \in P)$

implies if $(x)(\varepsilon_{m*<x>} \in P)$ then $\varepsilon_m \in P$

[since $\varepsilon_m<> = \varepsilon m$ and ②].

Thus $(m)((x)(\varepsilon_{m*<x>} \in P) \Rightarrow \varepsilon_m \in P)$.

Therefore by BI we have $(m)(\varepsilon_m \in P)$ and so in particular $\varepsilon = \varepsilon_{<>} \in P$ as required.

1.3.10. Remark. The proof of 1.3.9 is due to A.S. Troelstra in [18].

2. INTUITIONISTIC 2ND ORDER ARITHMETIC

2.1. LOGIC.

2.1.1. Remark. In considering logic before arithmetic we are at odds with the Intuitionists on two grounds. Firstly, though they concede logic is elementary in the sense that formalizing the idea of proof gives rise to different structures, logic is not elementary in that these structures are impredicative: having formalized the idea of proof, the questions that this formalizing was meant to answer, being tied up in the formalizing, are left unanswered. Secondly, Intuitionists regard their mathematics as constructive: that is, they start with simple constructions like natural numbers and build up more complex concepts by 'reflecting' on the properties of the constructions that are implicit in these concepts. In this sense statements in mathematics are not proved by applying logical rules previously agreed upon, but by applying constructions.

2.1.2. Remark. Constructions, then, are for the Intuitionists, the objects of mathematical research so that proofs are considered to be constructions. Notions are deducible properties of constructions. Three assumptions about notions are: if a and b are constructions then 'a is applicable to b' is a notion; if c is any construction and A is any statement, 'c is a proof of A' is a notion; if N(x is a notion with free variable x then 'a is a proof of N(x), for all x' is a notion. The last two assumptions are justified as follows: if we are in doubt as to whether a proves A then a does not prove A; we can mechanically check whether 'a is a proof of N(x) for all x'.

2.1.3. Remark. With these informal ideas of construction and notion the Intuitionists have given formal theories of constructions, e.g. G. Kreisel in Mathematical Logic in Lectures on modern mathematics III ed T.L. Saaty New York, 1965 p.95-195. These result in the following informal re-interpretation of the logical symbols:

A proof of $A \vee B$ is given by presenting a proof of A or a proof of B.

A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

A proof of $A \rightarrow B$ is given by a construction which transforms any given proof of A into a proof of B, together with a proof of this fact.

A proof of $\neg A$ is given by a proof of $A \rightarrow \perp$ where $\perp$ is some false statement like $0 = 1$.

A proof of $\exists x\, Ax$ is given by presenting a construction c and a proof of Ac.

A proof of $\forall x\, Ax$ is given by presenting a construction which to each object a, associates a proof Ac and a proof of this fact.

2.1.4. Remark. This leaves open the question of how proofs of atomic statements are given. For logic, Kreisel, in footnote 17 on p.159 in A survey of proof theory II in Proceedings of the second Scandanavian logic symposium, Oslo ed J.E. Fenstad, Amsterdam, North Holland 1971, points out this involves considering interpretations in all possible domains, whereas for arithmetic, for example, such proofs can be indicated e.g. the natural numbers are constructions obtained by juxtaposing units; 1, 11, 111, - etc. Hence, for Intuitionists, arithmetic is more basic than logic.

2.1.5. Remark. We give the following formal axiomatic system which yelds only those formulas which are provable by a formal scheme as mentioned in 2.1.3. This will be the 1st order system that our model will satisfy.

b is substitutable in A $=_{df}$ x does not occur free in A within the scope of $\forall y, \exists y$ for each y which occurs in b.

$S_x^b(A) =_{df} A$ iff b is not substitutable for x in A.

$S_x^b(A) =_{df}$ the formula obtained from A by replacing each free occurrence of x by an occurrence of b iff b is substitutable for x in A.

For the propositional calculus the axiom schemas are

$((A \to B) \to ((B \to C) \to (A \to C)))$ $\quad (A \to (A \vee B))$ $\quad (B \to (A \vee B))$

$((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$ $\quad ((A \wedge B) \to A)$ $\quad ((A \wedge B) \to B)$

$((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$ $\quad ((A \to (B \to C)) \to ((A \wedge B) \to C))$

$(((A \wedge B) \to C) \to (A \to (B \to C)))$ $\quad ((A \wedge \neg A) \to B)$ $\quad ((A \to (A \wedge \neg A)) \to \neg A)$.

For the predicate calculus the axiom schemas are

$\forall xA \to S_x^a(A)$ $\qquad S_x^a(A) \to \exists xA$.

The system is closed under the rules

if $A \to B$ then $A \to \forall yB$ where y does not occur free in A

if $A \to B$ then $\exists yA \to B$ where y does not occur free in B

if A and $A \to B$ then B.

2.1.6. Remark. As can be seen these differ from the axiom schemas of classical logic only in that intuitionistic logic is without $A \vee \neg A$. Hence intuitionistic logic retains the strong implication of classical logic but weakens the negation of classical logic. In view of the easily derived schemas $A \to (B \to A)$ and $\neg A \to (A \to B)$: informally if something is true then anything at all will imply it and if something is false then it implies anything at all; it can be argued that intuitionistic logic is not as constructive as is made out by the Intuitionists. However, it is not our task to sort out this argument.

2.2. SPECIES, MAPPINGS AND CONSTRUCTIVE (LAWLIKE) SEQUENCES.

2.2.1. Remark. Species may be regarded as the intuitionistic analogue of classical sets. Given a well-defined collection of mathematical objects, the well-defined properties of these collections are species. That a property P, associated with a collection A, is well-defined is at least meant to entail that we know what it means for Px to hold for $x \in A$, even if we do not know if Px holds. Thus species are not constructions in the sense that their elements are constructively given. Rather they are the results of applying a certain comprehension principle.

Predicative applications of this comprehension principle involve quantifications over elements of a basic collection only. So that Intuitionists see little objection against accepting all predicative applications of this comprehension principle. For us this means all arithmetical predicates of natural numbers define species.

Impredicative applications of this comprehension principle, which involve quantification over all subspecies of a basic collection, only concern us when we come to define K in 2.3. But to define K we need only a weak type of impredicativity, namely, a single generalized inductive definition.

In terms of the theory of constructions mentioned in 2.1.2, species correspond to notions which are themselves constructions.

<u>2.2.2. Remarks</u>. Even though species have been defined intensionally, Intuitionists retain an extensional equality between species, namely the usual classical relation $X = Y =_{df} (x)(x \in X \text{ iff } x \in Y)$.

Species are determined by applications of a comprehension principle, and so an element of a collection of mathematical objects is an 'element' of a property and hence a species, if it has or might have been defined independently of the property. Therefore different applications of this comprehension principle will give rise to different species which nevertheless have the same elements. Thus Intuitionists recognize this 'weaker' equality but do not regard it as primary as in classical set theory.

<u>2.2.3. Definition</u>. Let X and Y be species.

X is inhabited $=_{df} (Ex)(x \in X)$

X is empty $=_{df} \sim(Ex)(x \in X)$

Y is detachable in X $=_{df} (x \in X)(x \in Y \text{ or } x \notin Y)$

X is discrete $=_{df} (x \in X)(y \in X)(x = y \text{ or } x \neq y)$.

2.2.4. Remarks. Note that not (X is empty) does not imply X is inhabited, since $\neg\neg A \to A$ is invalid intuitionistic logic. Intuitionists are interested when $\neg\neg Px \to Px$ is valid and in this case such a predicate or relation P is said to be stable. To 'stabalize' equality, the Intuitionists introduce the apartness relation # on species as follows:
for each $x, y, z \in X$, X a species,
$\neg x \# y$ iff $x = y$, $x \# y$ iff $y \# x$, $x \# y$ iff $x \# z$ or $y \# z$.

The usual set theoretic operations e.g. $X \cup Y$, $X \times Y$ have their analogues in the theory of species. These analogues are defined analogously.

2.2.5. Definition. A mapping ϕ from a species X into a species Y is a process which assigns to each $x \in X$ an element $\phi x \in Y$ such that

$$x = x' \text{ implies } \phi x = \phi x' \qquad (*)$$

We write $\phi : X \to Y$ or $\phi \in (X)Y$ or $\phi \in Y^X$.

A mapping ϕ is bi-unique (1-1) $=_{df}$ for each $x, x' \in X$, $\phi x = \phi x'$ implies $x = x'$.
ϕ is weakly bi-unique $=_{df}$ for each $x, x' \in X$, $x \neq x'$ implies $\phi x \neq \phi x'$.

If $\phi \in (X)Y$ and X, Y have apartness relations #, #' then ϕ is strongly bi-unique $=_{df}$ for each $x, x \in X$, $x \# x'$ implies $\phi x \# \phi x'$.

2.2.6. Remark. Note that contrary to classical mathematics the motion of mapping is defined intensionally. The condition (*) is essential in case the equality relation in (*) is not intensional. For in this case, for example when the elements of X and Y are species, it must be shown that the extensional equality relation is preserved by the mapping.

If the equality relation on Y is stable then a weakly bi-unique mapping is bi-unique. A bi-unique mapping has an inverse denoted $\phi^{-1} \in (\phi[X])X$. If X has an apartness relation, ϕ is bi-unique and $\phi[X] = Y$, then ϕ induces an apartness relation on Y in the obvious way.

Notions such as homomorphism can be defined in the same way as the classical notions are defined.

<u>2.2.7. Remark</u>. For intuitionistic second order arithmetic the most important types of mappings are those whose form is (N)N or ((N)N)N or ((N)N)((N)N). Those of the first form are special cases of the more general class (N)X where X is a species. These are called sequences.

The simplest example of a sequence is that of lawlike or constructive sequence. These are sequences which are completely fixed in advance by a law e.g. primitive recursive functions. More correctly they are sequences given by an algorithm together with a proof of the applicability of the algorithm to all natural numbers. In this sense they have a complete description.

Lawlike sequences are intensionally equal iff they are given in the same way, so that extensional equality $(x)(\alpha x = \alpha' x)$ for α, α' sequences, does not imply intensional equality.

Because of their simplicity, existence of lawlike sequences is closely connected with choice principles. For example, if we have a proof of $(x)(Ey)A(x,y)$, where x, y range over N and A is a statement about complete objects and

A is extensional (i.e. $x = x'$ implies $Ax = Ax'$), then this proof must contain a complete description of how to associate a y to a given x. Thus there exists a lawlike sequence α with $(x)A(x, \alpha x)$ i.e. $(x)(Ey)A(x,y) \Rightarrow (E\alpha)(x)A(x, \alpha x)$ holds.

2.3. LAWLESS SEQUENCES OF NATURAL NUMBERS.

2.3.1. Remarks. Another notion of sequence, opposite to lawlike, is lawless. Informally these are sequences such that after constructing any initial segment we are able to choose freely any natural number to extend this segment. That is, there is no restriction on the choices at any time; this can be compared with the successive throws of a die. They have, then, a most incomplete description. Moreover the only dependence between two lawless sequences is intensional equality, so we cannot have such relations as $\alpha x = 2\beta x$.

We shall carry over the notation about sequences used in Chapter One and continue to make no distinction between an nni and the sequence assigned to it. Further we use $\equiv$ to indicate intensional equality and $=$ to indicate extensional equality, and $=_{df}$ as usual to indicate introduction by definition (which amounts to intensional equality).

2.3.2. Axiom. LS1 $(x)(E\alpha)(\alpha \in \{x\})$

2.3.3. Remark. LS1 states that any initial segment can be extended to a lawless sequence, which fits in with our informal interpretation. As a consequence, there are infinitely many extensions of an initial segment x, namely $x * \beta$, for any lawless β. As an example consider an initial segment x extended by a throw of a die.

2.3.4. Axiom. LS2 $(\alpha)(\beta)(\alpha \equiv \beta$ or $\alpha \not\equiv \beta)$

2.3.5. Remark. Two lawless sequences are intensionally equal if they are given by the same generating process. Therefore we know before we generate any initial segment of α or β whether we use the same generating process or not, i.e. whether α and β are the same sequence or not.

2.3.6. Axiom. LS3 $\not\equiv(\alpha, \alpha_0, \ldots, \alpha_k)$ & $A(\alpha, \alpha_0, \ldots, \alpha_k)$

$\Rightarrow (En)(\alpha \in \{n\}$ & $(\beta \in \{n\})(\not\equiv(\beta, \alpha_0, \ldots, \alpha_k) \Rightarrow A(\beta, \alpha_0, \ldots \alpha_k)))$

where $\not\equiv(\alpha, \alpha_0, \ldots, \alpha_k) =_{df} (\alpha \not\equiv \alpha_0$ & ... & $\alpha \not\equiv \alpha_k)$ and A is an extensional statement about x.

2.3.7. Remark. We argue for LS3 as follows. Given $\not\equiv(\alpha, \alpha_0, \ldots, \alpha_k)$ and $A(\alpha, \alpha_0, \ldots, \alpha_k)$, because α is lawless and $\not\equiv(\alpha, \alpha_0, \ldots, \alpha_k)$, nothing about $\alpha_0, \ldots, \alpha_k$ helps to generate α, so that $A(\alpha, \alpha_0, \ldots, \alpha_k)$ holds because $(Ex)A(\bar{\alpha}x, \alpha_0, \ldots, \alpha_k)$ holds. Hence if β is lawless and $\not\equiv(\beta, \alpha_0, \ldots, \alpha_k)$ and $\beta \in \{\bar{\alpha}x\}$ then $A(\beta, \alpha_0, \ldots, \alpha_k)$ holds.

We need the condition $\not\equiv(\alpha, \alpha_0, \ldots, \alpha_k)$, otherwise we obtain a contradiction by applying LS3 to $A(\alpha, \gamma) =_{df} \alpha \equiv \gamma$ and then replace γ with α.

2.3.8. Proposition. For each lawless α, β, $\alpha \equiv \beta$ iff $(x)(\alpha x = \beta x)$.

Proof. Left to right is obvious. From right to left we argue by contradiction. $\alpha \not\equiv \beta$ and $(x)(\alpha x = \beta x)$ and LS3 implies $(En)(\alpha \in \{n\}$ & $(\gamma \in \{n\})(\gamma \not\equiv \beta \Rightarrow (x)(\gamma x = \beta x)))$. LS1 implies there is a γ_1 with $\gamma_1 \in \{n * \langle \beta(lhn) + 1 \rangle\}$.

$\gamma_1 \not\equiv \beta$ and $\gamma_1 \in \{n\}$ implies $(x)(\gamma_1 x = \beta x)$, a contradiction. Hence $\sim \alpha \not\equiv \beta$ so by LS2 $\alpha \equiv \beta$.

2.3.9. Remark. As a corollary of 2.3.8 we have $\alpha = \beta$ or $\alpha \neq \beta$, which is surprising considering the lawless of α and β.

As a special case of LS3 we have $A\alpha \Rightarrow (En)(\alpha \in \{n\}\ \&\ (\beta \in \{n\})A\beta)$, from which we can easily derive the weak continuity property for natural numbers:

WC-N $(\alpha)(Ex)A(\alpha, x) \Rightarrow (\alpha)(Ex)(Ey)(\beta)(\bar{\beta}y = \bar{\alpha}y \Rightarrow A(\beta, x))$.

This is called a continuity property since, as mentioned before, if $A(\alpha, x)$ is the statement $\Phi\alpha = x$ for functional $\Phi : N^N \to N$ with the Baire Space topology on N^N and the discrete topology on N, then WC-N says Φ is continuous.

2.3.10. Remark. As a consequence of LS3 (by applying LS3 to suitable statements about lawless sequences) we have the following properties of lawless sequences:

(1) $(\alpha)\sim\sim(Ex)(\alpha x = 0)$ (2) $(\alpha)\sim(E\beta)(x)(\beta x = \alpha x + 1)$

(3) $\sim(E\alpha)(Ea)(\alpha = a)$ where a is a lawlike sequence.

(2) shows that lawless sequences are not closed under even trivial relationships. In fact they are only closed under identity relations or mappings. (3) shows that lawless sequences are provably non-lawlike.

For these two reasons they are not directly useful for studying second-order arithmetic. However they are used to show the completeness of intuitionistic logic and are used to refute classically valid statements. Furthermore A.S. Troelstra in [19] has used them to construct a model for choice sequences but this is not our model.

2.4. CONTINUITY AND BROUWER OPERATIONS

2.4.1. Remarks. In this section we want to consider functionals, specifically certain continuous functionals, from N^N into N and N^N into N^N. We are not, until the last two remarks, concerned as to what type of sequence we are talking about, as these results will hold for all sequences.

The way we introduce these continuous functionals is through Brouwer operations which we shall see can be intepreted as nbdfns on Baire Space.

Intuitively a continuous functional $\Phi : N^N \to N$ can be thought of as an operator which determines the value for a given argument on just an initial segment of the argument, cf. $(\alpha)(x)(\Phi\alpha = x \Rightarrow (En)(\alpha \in \{n\}\ \&\ (\beta)(\beta \in \{n\} \Rightarrow \Phi\beta = x)))$. Hence we try to replace the functional Φ with an operator which is defined on finite sequences, i.e. Baire Space neighbourhoods.

In this next definition, we use the notion of function abstraction (commonly called Church's λ-operator). Whereas quantifiers attach to formulas to produce formulas, function abstraction attaches to terms to produce terms. Suppose '...x...' stands for term containing 'x' as a free variable. $\lambda x(\ldots x\ldots)$ is the function whose value for each argument x is ...x... .

2.4.2. Definition. We inductively define a class K of constructive or lawlike functions as follows, writing Ka for $a \in K$, $a_{\langle x\rangle}n$ for $a(\langle x\rangle * n)$.

K1 $a = \lambda n \cdot x^+ \Rightarrow Ka$ K2 $a\langle\rangle = 0\ \&\ (x)K(\lambda n \cdot a_{\langle x\rangle}n) \Rightarrow Ka.$

and K is the least such class.

Alternatively if we define $A_K(Q, a)$ as follows:
$A_K(Q,a) =_{df} (Ey)(a = \lambda x\cdot y^+)$ or $(a\langle\rangle = 0 \;\&\; (x)Q(\lambda n\cdot a_{\langle x\rangle}n))$
for each class Q; then K1, K2 for Q is $A_K(Q,a) \Rightarrow Qa$ and the minimality of K is K3 $(a)(A_K(Q,a) \Rightarrow Qa) \Rightarrow (a)(Ka \Rightarrow Qa)$.
Elements of K will be denoted by e, f, with or without subscripts, and are called Brouwer operations.

2.4.3. Remark. From the inductive definition of K we can derive the principle of induction over unsecured sequences:
$(a \in K)[((n)(an \neq 0 \Rightarrow Qn) \;\&\; (n)((y)Q(n * \langle y\rangle) \Rightarrow Qn)) \Rightarrow Q0]$ ⊛.
Define $Aa =_{df} (m)[[(n)(an \neq 0 \Rightarrow Q(m*n)) \;\&\; (n)((y)Q(m*n*\langle y\rangle)$

$$\Rightarrow Q(m*n))] \Rightarrow Qm].$$

We shall show $(a \in K)Aa$. For $a = \lambda x \cdot y^+$ Aa is immediate.
Assume (i) $a\langle\rangle$ and $(x)A(\lambda n\cdot a_{\langle x\rangle}n)$,

(ii) $(n)(an \neq 0 \Rightarrow Q(m*n))$,

(iii) $(n)((y)Q(m*n*\langle y\rangle) \Rightarrow Q(m*n))$.

From (ii) it follows that

(iv) $(x)(n)(a_{\langle x\rangle}n \neq 0 \Rightarrow Q(m*\langle x\rangle*n)$.

From (iii) it follows that

(v) $(x)(n)((y)Q(m*\langle x\rangle*n*\langle y\rangle) \Rightarrow Q(m*\langle x\rangle*n))$,

then by (i), (iv), (v) it follows that $(x)Q(m*\langle x\rangle)$.
Applying (iii) with $n = 0$ gives Qm.
Therefore by K3, $(a \in K)Aa$, so as a particular case we have ⊛.

2.4.4. Proposition. $(e \in K)(\alpha)(Ex)(e\bar{\alpha}x \neq 0)$ ①
$(e \in K)(m)(n)(em \neq 0 \Rightarrow e(m*n) = en)$ ②.

Proof. By the inductive definition of K, for example ①.
Define Q as follows, $Qe =_{df} (\alpha)(Ex)(e\bar{\alpha}x \neq 0)$.
Obviously $Q(\lambda n\cdot y^+)$ holds. Given e suppose $e\langle\rangle = 0$ and $(y)(Q(\lambda n\cdot e_{\langle y\rangle}n)$.

Given α with $\alpha = \langle\alpha 0\rangle * \alpha'$, by assumption $Q(\lambda n \cdot e_{\langle\alpha 0\rangle} n)$. It follows that $(Ey)(e_{\langle\alpha 0\rangle}\bar{\alpha}'y \neq 0)$. Therefore $(Ey)(e\bar{\alpha}y^{+} \neq 0)$. Hence Qe holds so that $(e \in K)(A_K(a,e) \Rightarrow Qe)$. So by K3 $(e \in K)Qe$.

<u>2.4.5. Proposition</u>. $(\alpha)(E!\,x)(Ey)(e\bar{\alpha}y = x^{+})$.

<u>Proof</u>. from 2.4.4.

<u>2.4.6. Definition</u>. By 2.4.5 we can unambiguously define a functional $\Phi^e : N^N \to N$ as follows, $\Phi^e(\alpha) =_{df} x$ iff $(Ey)(e\bar{\alpha}y = x^{+})$. In short we write $e(\alpha)$ for $\Phi^e(\alpha)$, $K^* =_{df} \{e(\alpha) : e \in K\}$.

<u>2.4.7. Remark</u>. In 1.3 we showed that $BI \Rightarrow K = Nfn$. Intuitionistically, by a very similar argument, we can show $BI \Rightarrow K = \{e : (\alpha)(Ex)(e\bar{\alpha}x \neq 0)\ \&\ (m)(n)(em \neq 0 \Rightarrow e(m*n) \neq 0)\}$. It is not difficult to see that Nfn is just $\{e : (\alpha)(Ex)(e\bar{\alpha}x \neq 0)\ \&\ (m)(n)(em \neq 0 \Rightarrow e(m*n) \neq 0)\}$ so that we can interpret the Brouwer operations as nbdfns and the Φ^e's as Φ^E's.

<u>2.4.8. Proposition</u>. $(\alpha)(E!\beta)(\beta x = y$ iff $(Ez)(e_{\langle x\rangle}\bar{\alpha}z = y^{+}))$.

<u>Proof</u>. By using the inductive definition of K as in 2.4.3, 2.4.4.

<u>2.4.9. Definition</u>. Using 2.4.8 we can unambiguously define functionals $\Phi^e : N^N \to N^N$ as follows:
$\Psi^e(\alpha) =_{df} \beta$ iff $(\beta x = y$ iff $(Ez)(e_{\langle x\rangle}\bar{\alpha}z = y^{+}))$. In short we write $e|\alpha$ for $\Psi^e(\alpha)$, $K^{**} =_{df} \{e|\alpha : e \in K\}$.

<u>2.4.10. Remarks</u>. We can define Ψ^e equivalently by:
$\Psi^e(\alpha)(y) =_{df} x$ iff $(Ez)(e_{\langle y\rangle}\bar{\alpha}z = x^{+})$.

As was remarked in 1.2, the Φ^{e}'s , Ψ^{e}'s are continuous. However we cannot use the proof of 1.2.6 to show that all continuous functionals from N^{N} into N or N^{N} are of the form Φ^{e} or Ψ^{e} for some e. We now give an assumption on lawless sequences which allows us to show something stronger: namely that all functionals from N^{N} into N or N^{N} are of the form Φ^{e} or Ψ^{e} for some e. We do this by strengthening WC-N.

2.4.11. Axiom. LS4 $(\alpha_0)\ldots(\alpha_p)(\#(\alpha_0,\ldots,\alpha_p)$
$\Rightarrow (Ex)A(\alpha_0,\ldots,\alpha_p, x)) \Rightarrow (Ee)(\alpha_0)\ldots(\alpha_p)(\#(\alpha_0,\ldots,\alpha_p)$
$\Rightarrow A(\alpha_0,\ldots,\alpha_p, e(\lambda x . < \alpha_0 x \ldots \alpha_p x>)))$

2.4.12. Remark. A special case of LS4 is the principle of bar continuity BC-N, $(\alpha)(Ex)A(\alpha,x) \Rightarrow (Ee)(\alpha)A(\alpha, e(\alpha))$, from which with $A(\alpha,x) =_{df} \Phi\alpha = x$, we have $(Ee)(\alpha)[\Phi\alpha = e(\alpha)]$ for each Φ.

2.5. CHOICE SEQUENCES.

2.5.1. Remark. Between the notion of lawlike and lawless sequences is the notion of choice sequence. Informally a choice sequence is a sequence for which an initial segment is known and for which the following terms though unknown are known to lie within certain bounds which may be decreasing. This process may be considered as generating pairs $<x, R>$ of a natural number and a condition which restricts future values. Hence the condition is extensional. Moreover as the conditions may become more restrictive we have some relations between conditions expressing this restrictiveness. Thus the notions of lawlike and lawless sequences may be regarded as limiting cases of choice sequences: on the one hand choice sequences with the most restrictive conditions (bounds which are equal) and on the other choice sequences with no conditions.

These informal ideas, when formalized, give rise to several notions of choice sequence. We shall consider choice sequences generated by continuous operations which are due to A.S. Troelstra in [18].

2.5.2. Remark. The motivation for Troelstra's notion of choice sequence is the following: when speaking about an arbitrary number we often specify x is such that for some y, $x = f(y)$, and perhaps later specify y is such that for some z, $y = g(z)$, where f and g are lawlike functions.

So a choice sequence α is a sequence of pairs $\langle\langle x_n, R_n\rangle_n\rangle$, where x_n is a natural number and R_n is an element of a class $\mathcal{R}$ which contains all conditions of the form $R =_{df} \lambda\alpha\cdot(\alpha = \Gamma_0\alpha_0 \;\&\; \alpha_0 = \Gamma_1\alpha_1 \;\&\ldots\&\; \alpha_{n-1} = \Gamma_n\alpha_n)$, where the α, α_i are choice sequences and the Γ_is are continuous operators from N^N into N^N. Moreover the R_n in the members of a choice sequence satisfy the following relation C : if $R_n = \lambda\alpha\cdot(\alpha = \Gamma_0\alpha_0 \;\&\ldots\&\; \alpha_{n-1} = \Gamma_n\alpha_n)$ then $R_{n+1} =_{df} \lambda\alpha\cdot\ (\alpha = \Gamma_0\alpha_0 \;\&\ldots\&\; \alpha_{n-1} = \Gamma_n\alpha_n \;\&\; \alpha_n = \Gamma_{n+1}\alpha_{n+1}$ $\&\ldots\&\; \alpha_{n+p-1} = \Gamma_{n+p}\alpha_{n+p})$ for some p.

However this is too simple an idea, for a choice sequence may depend on more than one other sequence at any time. We can easily extend the class $\mathcal{R}$ to allow for this by including all conditions of the form:

$R =_{df} \lambda\alpha\cdot[\alpha = \Gamma_0(\lambda x\cdot(\alpha_0 x, (\alpha_1 x, \ldots(\alpha_{m-1}x, \alpha_m x)\ldots)))$ &

$$\mathop{\&}_{i=1}^{m} \alpha_0 = \Gamma_i(\lambda x . (\alpha_{i1}x, (\alpha_{i2}x, \ldots (\alpha_{im_i-1}, \alpha_{im_i})\ldots))) \ \&$$

$$\mathop{\&}_{i=1}^{m} \mathop{\&}_{j=1}^{m} \alpha_{ij} = \Gamma_{ij}(\lambda x. (\alpha_{ij1}x, (\alpha_{ij2}x, \ldots (\alpha_{ijm_i-1}, \alpha_{ijm_i})\ldots)))\ldots]$$

where $(,)$ is the pairing function of 1.1.5. But such conditions merely obscure in a mess of symbols the principle behind choice sequences generated by continuous operations, so we confine ourselves to looking at choice sequences with simpler conditions which in any case can in principle be extended to choice sequences with the complicated conditions.

From this description of choice sequences we have this important fact.

2.5.3. Proposition. Choice sequences are closed under continuous operations.

Proof. Suppose $\alpha = \ll\alpha x, R_x^{\alpha}>x>$ with $R_0^{\alpha} =_{df} \lambda\alpha . [\alpha = \Gamma_0\alpha_0]$, $R_1^{\alpha} =_{df} \lambda\alpha . [\alpha = \Gamma_0\alpha_0 \ \& \ \alpha_0 = \Gamma_1\alpha_1 \ \&\ldots\& \ \alpha_p = \Gamma_{p+1}\alpha_{p+1}]$ etc, and suppose $\beta = \Gamma\alpha$ where Γ is continuous.
Then $\beta =_{df} \ll\beta 0, R_0^{\beta}>,\ldots>$ with $R_0^{\beta} =_{df} \lambda\beta . [\beta = \Gamma\alpha \ \& \ \alpha = \Gamma_0\alpha_0]$. Then $R_0^{\beta} \in \mathcal{R}$, since Γ is continuous, and we can extend β by any R with $C(R_0^{\beta}, R)$. Thus β is a choice sequence.

2.5.4. Remark. We can get a relationship, for extensional statements X about choice sequences (i.e. if $X\alpha$ and $\alpha = \beta$ then $X\beta$), between choice sequences, namely
$X\alpha$ iff $(E\Gamma)(E\beta)(\alpha = \Gamma\beta) \ \& \ (\beta)X(\Gamma\beta)$, which is called intensional continuity.

Right to left is trivial. If we can assert $X\alpha$ then this must be on account of some initial segment of α, say

$\ll x_0, R_0^\alpha >, \ldots, < x_n, R_n^\alpha \gg$.

Suppose $R_n^\alpha =_{df} \lambda\alpha . (\alpha = \Gamma_1\alpha_1 \& \ldots \& \alpha_n = \Gamma_{n+1}\alpha_{n+1})$. If α_{n+1} is completely undetermined then it is arbitrary, as is $(\beta)X(\Gamma_1 \ldots \Gamma_{n+1}\beta)$ since X is extensional. If α_{n+1} is determined up to $<v_0, \ldots, v_x>$ then we can consider $\alpha_{n+1} = \Gamma_{n+2}\alpha_{n+2}$, where α_{n+2} is any arbitrary sequence and Γ_{n+2} is defined by

$(\alpha)(y)((y \leqslant x \Rightarrow \Gamma_{n+2}(\alpha)(y) = v_y) \& (y > x \Rightarrow \Gamma_{n+2}(\alpha)(y) = \alpha y))$

So $\alpha = \Gamma_1 \ldots \Gamma_{n+2}\alpha_{n+2}$. Hence $(\beta)X(\Gamma_1 \ldots \Gamma_{n+2}\beta)$.

2.5.5. Remark. The principle of extensional continuity is what for lawless sequences we called WC-N, i.e. $(\alpha)(Ex)X(\alpha,x) \Rightarrow (\alpha)(Ex)(Ey)(\beta)(\bar{\alpha}y = \bar{\beta}y \Rightarrow X(\beta,y))$. For the various notions of choice sequence this has been justified by the principle of intensional continuity. Through considering this principle, A.S. Troelstra in [18] developed an abstraction operator from which he is able to prove WC-N for our notion of choice sequence. This abstraction operator does away with the condition part of choice sequences. Given a choice sequence $\alpha = \ll x_n, R_n^\alpha > n >$, Abstr $\alpha =_{df} \ll x_n, U>_n>$, where U is the universal condition $\lambda\alpha . \alpha = \alpha$. Thus we forget the manner in which α is generated. Hence we cannot argue $\alpha \equiv$ Abstr α. For any term t, it is effectively verifiable that $\bar{\alpha}t \equiv \overline{(\text{Abstr } \alpha)}t$, but we cannot assert $\bar{\alpha}x \equiv \overline{(\text{Abstr } \alpha)}x$ and then use the rule of generalization to give $(x)(\bar{\alpha}x \equiv \overline{(\text{Abstr } \alpha)}x)$, for this would imply $\alpha \equiv$ Abstr α.

This breakdown in the rule of generalization means that attempts to formalize Abstr α are fraught with difficulties. Furthermore it is not surprising that such an unusual operator as Abstr allows us to derive WC-N, since even Intuitionists will agree that a contradiction will imply anything at all. But for another reason, not connected with an attempt to formalize such Abstraction operators, we find any justification of WC-N hard to accept. This is that it can be shown intuitionistically that WC-N and AC-NF($(x)(E\beta)A(x,\beta) \Rightarrow (E\alpha)(x)A(x,\alpha_x)$ where $\alpha_x(y) =_{df} \alpha(x,y)$, (,) the pairing function) and BI imply BC-N, which as we remarked before implies that all functionals from N^N into N are continuous - a classically invalid result.

2.5.6. Remark. Recall from 2.4.12 that BC-N is $(\alpha)(Ex)A(\alpha,x) \Rightarrow (Ee)(\alpha)A(\alpha,e(\alpha))$. While this does not hold in classical mathematics we will show that a special case of BC-N called special bar continuity does hold. Previously we have not spoken much of choice principles. In Chapter One we used a choice principle and the law of excluded middle to show 1.2.6; this is a standard classical technique, which, since the law of excluded middle does not hold, is inapplicable to intuitionistic mathematics. For this reason choice principles are not as useful for Intuitionists.

The simplest choice principle for sequences is AC-NN $(x)(Ey)A(x,y) \Rightarrow (E\alpha)(x)A(x,\alpha x)$, which is classically equivalent to WF(R) $\Rightarrow$ TI(R) and equivalent to BI. However even AC-NF (mentioned in 2.5.4) does not imply BI, so that interesting results come from an application of both these principles.

For the next proposition, which is due to G. Kreisel and A.S. Troelstra in [11], we are using SBC for

$(\alpha)(Ex)A(\bar{\alpha}x) \Rightarrow (E\varepsilon \in Nfn)(n)(\varepsilon n \neq 0 \Rightarrow (\beta)(\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon n)^{-})))$.

We show that SBC is classically derivable from BI and AC-NF.

<u>2.5.7. Proposition.</u> BI & AC-NF $\Rightarrow$ SBC.

<u>Proof.</u> We suppose $(\alpha)(Ex)A(\bar{\alpha}x)$ and use BI with the 'barring' statement $(Ex)(\beta)[\beta \in \{m\} \Rightarrow A(\bar{\beta}x)]$ and the 'inducting' statement $(E\varepsilon^m \in Nfn)(\beta)[\beta \in \{m\} \Rightarrow (n)[\varepsilon^m n \neq 0 \Rightarrow [\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^m n)^{-})]]]$ about m. Then

$(\alpha)(Ex)A(\bar{\alpha}x)$ implies $(\alpha)(Ex)(\beta)[\beta \in \{\bar{\alpha}x\} \Rightarrow A(\bar{\beta}x)]$

[since $\beta \in \{\bar{\alpha}x\}$ implies $\bar{\beta}x = \bar{\alpha}x$]

implies $(\alpha)(Ex)(Ey)(\beta)[\beta \in \{\bar{\alpha}x\} \Rightarrow A(\bar{\beta}y)]$

[take y to be x].

Since for each m, n, $\beta \in \{m*n\}$ implies $\beta \in \{m\}$ we have

$(m)(n)((Ex)(\beta)(\beta \in \{m\} \Rightarrow A(\bar{\beta}x)) \Rightarrow (Ex)(\beta)(\beta \in \{m*n\} \Rightarrow A(\bar{\beta}x)))$.

These are the first and second hypotheses for BI.

Suppose given m, $(Ex)(\beta)(\beta \in \{m\} \Rightarrow A(\bar{\beta}x))$.

Define a sequence ε^m as follows:

$\varepsilon^m n =_{df} 0$ iff $lhn < lhm$

$\varepsilon^m n =_{df} x^+$ iff $lhn \geq lhm$, where x is the least x such that $(\beta)(\beta \in \{m\} \Rightarrow A(\bar{\beta}x))$.

For each α, $\varepsilon^m(\bar{\alpha}y) = x^+$ where $y = (lhm)^{-}$.

For each m', n, x, $\varepsilon^m m' = x^+$ implies $lhm' \geq lhm$

implies $lh(m'*n) \geq lhm$

implies $\varepsilon^m(m'*n) = x^+$.

Therefore $\varepsilon^m \in Nfn$ and

$(Ex)(\beta)(\beta \in \{m\} \Rightarrow A(\bar{\beta}x))$

implies $(Ex)(\beta)[\beta \in \{m\} \Rightarrow (n)(\varepsilon^m n \neq 0 \Rightarrow (\varepsilon^m n = x^+ \;\&\; A(\bar{\beta}x)))]$

implies $(Ex)(\beta)[\beta \in \{m\} \Rightarrow (n)(\varepsilon^m n \neq 0 \Rightarrow A(\bar{\beta}(\varepsilon^m n)^{-}))]$

implies $(\beta)[\beta \in \{m\} \Rightarrow (n)(\varepsilon^m n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^m n)^-)))]$

[since x does not occur in the statement and

$\beta \in \{n\}$ is redundant].

Thus we have $(m)[(Ex)(\beta)(\beta \in \{m\} \Rightarrow A(\bar{\beta}x)) \Rightarrow$

$(E\varepsilon^m \in Nfn)(\beta)(\beta \in \{m\} \Rightarrow (n)(\varepsilon^m n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^m n)^-))))$,

and this is the third hypothesis of BI.

Let us denote the 'inducting' statement by $(E\varepsilon^m \in Nfn)Bm$. For the fourth hypothesis of BI, we suppose for given m that $(x)(E\varepsilon^{m*\langle x\rangle} \in Nfn)B(m*\langle x\rangle)$, which by AC-NF implies $(E\delta)(x)(\delta_x \in Nfn \;\&\; B'x)$, where $B'x$ is $B(m*\langle x\rangle)$, with δ_x replacing $\varepsilon^{m*\langle x\rangle}$. Define a sequence ε^m as follows:

$\varepsilon^m n =_{df} 0$ iff $lhn \leqslant lhm$; $\varepsilon^m n = \delta_z n$ iff $lhn > lhm$, where $z = n(lhm)$.

This ε^m satisfies the conditions for being in Nfn as follows:

For each $\alpha, y \geqslant (lhm)^+$ implies $\varepsilon^m(\bar{\alpha}y) = \delta_z(\bar{\alpha}y)$

where $z = (\bar{\alpha}y)(lhm) = \alpha(lhm)$

implies $\varepsilon^m(\bar{\alpha}y) = \delta_{\alpha(lhm)}(\bar{\alpha}y) \in Nfn$.

For each m', n, x,

$\varepsilon^m m' = x^+$ implies $lhm' > lhm$ and $\delta_z m' = x^+$, where $z = m'(lhn)$

implies $lhm' > lhm$ and $\varepsilon^m(m'*n) = \delta_{z'}(m'*n)$

where $z' = (m*n)(lhm)$

[since $lh(m'*n) \geqslant lhm' > lhm$]

implies $\varepsilon^m(m'*n) = \delta_z(m'*n)$

[since if $lhm' > lhm$ then $(m'*n)(lhm) = m'(lhm)$]

implies $\varepsilon^m(m'*n) = \delta_z(m'*n) = x^+$

[since $\delta_z \in Nfn$].

Now we have the conclusion of the fourth hypothesis for BI, for each β with $\beta \in \{m\}$ and for each n,

$\varepsilon^m n \neq 0$ and $\beta \in \{n\}$

implies $lhn > lhm$ and $\varepsilon^m n = \delta_z n \neq 0$, where $z = n(lhm)$

implies $m \leqslant n$ and $\varepsilon^m n = \delta_z n \neq 0$, where $z = n(lhm)$

[since $\beta \in \{m\}$ and $lhm < lhn$ and $\beta \in \{n\}$]

implies $m{*}\langle z\rangle \leqslant n$ and $\varepsilon^m n = \delta_z n \neq 0$ and $\beta \in \{n\}$

implies $\varepsilon^m n = \delta_z n$ and ($\beta \in \{m{*}\langle z\rangle\}$ and $\delta_z n \neq 0$ and $\beta \in \{n\}$)

implies $\varepsilon^m n = \delta_z n$ and $A(\bar{\beta}(\delta_z n)^-)$

[since we know that δ_z satisfies $B'x$ which is

$(\beta)(\beta \in \{m\} \Rightarrow (n)(\delta_x n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\delta_x n)^-))))$

and we have the three antecedents]

implies $A(\bar{\beta}(\varepsilon^m n)^-)$.

Therefore we have for this $\varepsilon^m \in Nfn$,

$$(\beta)(\beta \in \{m\} \Rightarrow (n)((\varepsilon^m n \neq 0 \ \&\ \beta \in \{n\}) \Rightarrow A(\bar{\beta}(\varepsilon^m n)^-))),$$

which is equivalent to

$(\beta)(\beta \in \{m\} \Rightarrow (n)((\varepsilon^m n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^m n)^-))))$.

Hence $(m)(E\varepsilon^m \in Nfn)(\beta)(\beta \in \{m\} \Rightarrow (n)(\varepsilon^m n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^m n)^-))))$ by BI. So in particular with $m = \langle\rangle$,

$(E\varepsilon^{\langle\rangle} \in Nfn)(\beta)(\beta \in \{\langle\rangle\} \Rightarrow (n)(\varepsilon^{\langle\rangle} n \neq 0 \Rightarrow (\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon^{\langle\rangle} n)^-))))$.

Thus since $\varepsilon^{\langle\rangle} = \varepsilon$ and $(\beta)(\beta \in \{\langle\rangle\})$ we have

$(E\varepsilon \in Nfn)(n)(\varepsilon n \neq 0 \Rightarrow (\beta)(\beta \in \{n\} \Rightarrow A(\bar{\beta}(\varepsilon n)^-)))$.

2.5.8. Remark. It can be shown intuitionistically that 2.5.7 holds, where we substitute $e \in K$ for $\varepsilon \in Nfn$ in SBC. The proof is very similar to that of 2.5.7. AC-NF is used to construct an e which by a similar method is shown to be an element of K.

We next show that SBC together with the inductive definition of K implies BI, so that we have the classical equivalence of $WF(R) \Rightarrow TI(R)$; BI and SCB & K in the presence of AC-NF.

The proof is due to G. Kreisel and A.S. Troelstra in [11].

2.5.9. Proposition. SBC & K $\Rightarrow$ BI.

Proof. From SBC and the hypothesis $(\alpha)(Ex)A(\bar{\alpha}x)$ of BI we conclude

$$(Ee)(n)(en \neq 0 \Rightarrow (\alpha)(\alpha \in \{n\} \Rightarrow A(\bar{\alpha}(en)^{-}))) \quad (1).$$

If $lhn \geqslant (en)^{-}$ then $n = \bar{\alpha}(en)^{-} * m$ for any $\alpha \in \{n\}$ and for some m. So by the hypothesis $(m)(n)(Am \Rightarrow A(m*n))$ of BI we have from (1),

$$en \neq 0 \Rightarrow An.$$

Hence by the hypothesis $(m)(Am \Rightarrow Bm)$ of BI we have

$$en \neq 0 \Rightarrow Bn.$$

If $lhn < (en)^{-}$ then $\bar{\alpha}(en)^{-} = n * m$ for any $\alpha \in \{n\}$ and some m. So by the hypothesis $(m)(Am \Rightarrow Bm)$ of BI we have

$$en \neq 0 \Rightarrow (\alpha)(\alpha \in \{n\} \Rightarrow B(\alpha(en)^{-}))$$

so that $en \neq 0 \Rightarrow (\alpha)(\alpha \in \{n\} \Rightarrow (m)B(n * m))$

$$\Rightarrow (x_1)\ldots(x_{lhm})B(n * \langle x_1\rangle * \ldots * \langle x_{lhm}\rangle).$$

So using hypothesis $(m)((y)B(m * \langle y\rangle) \Rightarrow Bm)$ lhm times we get

$$en \neq 0 \Rightarrow Bn.$$

Hence $(n)(en \neq 0 \Rightarrow Bn)$. (2).

(2), the hypothesis $(m)((y)B(m * \langle y\rangle) \Rightarrow Bm)$ of BI and the principle of induction over unsecured sequences implies $B\langle\rangle$.

As we saw in 1.3.6 the BI with conclusion $B\langle\rangle$ is equivalent to the BI with conclusion $(m)Bm$, so BI holds.

2.5.10. Remark. The formal system for choice sequences for which we shall give a topological model is a modification of the Kleene-Vesley system. Apart from the logical axiom schemas for the 1st and 2nd order logic and the usual arithmetic axiom schemas, the other axiom schemas are BI!, AC-NF and a special case of Brouwer's Principle for functions, namely,

$$\circledast (\alpha)(E!\beta)A(\alpha,\beta) \Rightarrow (E\tau)(\alpha)((x)(E!y)(\tau(\langle x\rangle * \bar{\alpha}y) > 0) \ \& \ (\beta)((x)(Ey)(\tau(\langle x\rangle * \bar{\alpha}y) = \beta(x) + 1 \Rightarrow A(\alpha,\beta)))$$

This schema with (E!β) replaced by (Eβ) has been shown by S.C. Kleene in [10] to imply Brouwer's Principle for numbers which we call BC-N.

So while we have attempted to give a classical justification for BI and so BI!, through its equivalence with WF(R) $\Rightarrow$ TI(R) and more importantly with SBC and K, in view of 2.4.12 we can give no justification for $\circledast$. The most we can do is say $\circledast$ is a restricted generalization of SBC which is classically valid, and in the simpler case of lawless sequences of the end of 2.4 we found BC-N held.

3. TOPOLOGICAL MODELS OF INTUITIONISTIC LOGIC

3.0.1. Remark. In this chapter we shall show why intuitionistic logic gives rise to topological models. In the first three sections we shall give the necessary characterization and representation theorems for topological Boolean algebras and psendo-Boolean algebras which will be used in the fourth section to show that intuitionist first order logic with equality has topological models.

3.1. BOOLEAN ALGEBRAS.

3.1.1. Remarks. Recall that an ordering in a set $\mathbf{A}$ is a lattice ordering when for each $a,b \in \mathbf{A}$, $\text{lub}(a,b) = a \cup b$, $\text{glb}(a,b) = a \cap b$ exist. Such an ordered set is called a lattice, and $a \cup b$ the join of a and b, $a \cap b$ the meet of a and b. A subset $\mathbf{A}'$ of lattice $\mathbf{A}$ is a sublattice iff it is closed under join and meet. A function from lattice $\mathbf{A}$ into lattice $\mathbf{B}$ which preserves join and meet is called a lattice homomorphism. We then define lattice mono-, epi- and isomorphisms in the usual way. Note that as the join and meet are uniquely determined by the lattice ordering, a bijective mapping h is an isomorphism iff ($h(a) \leqslant h(b)$ iff $a \leqslant b$).

Recall that a lattice is distributive iff for all $a,b \in \mathbf{A}$, $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ and $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$. In fact in view of duality arguments we need only see that one of these conditions is satisfied.

Recall that if a lattice $\mathbf{A}$ has a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$, then we can define $\cap$-complement and $\cup$-complement as follows: $c \in \mathbf{A}$ is the $\cap$-complement of

$a \in A =_{df}$ c is the greatest element with $a \cap c = 0$. $c \in A$ is the $\cup$-complement of $a \in A =_{df}$ c is the least element with $a \cup c = 1$. The complement of $a \in A$ is defined as the element which is simultaneously the $\cap$-complement and $\cup$-complement of a.

A distributive lattice such that each element has a complement is called a Boolean algebra.

<u>3.1.2. Definition</u>. A non-empty set ∇ of elements of a lattice A is a filter of $A =_{df}$ for each $a,b \in A$, $a \cap b \in \nabla$ iff $a \in \nabla$ and $b \in \nabla$. This condition is equivalent to ($a,b \in \nabla$ implies $a \cap b \in \nabla$) and ($a \in \nabla$ and $a \leqslant b$ implies $b \in \nabla$). The dual notion of a filter is that of ideal denoted by Δ.

A filter is maximal in $A =_{df}$ it is proper and not a proper subset of any proper filter in A.

A filter is prime in $A =_{df}$ it is proper and $a \cup b \in \nabla$ implies either $a \in \nabla$ or $b \in \nabla$.

<u>3.1.3. Remark</u>. Given any non-empty subset A' of a lattice A, the filter that is the intersection of all filters containing A' is called the filter generated by set A'. If A has 1 then the hypothesis that A' is non-empty can be dropped.

Via this idea we can show that every non-degenerate lattice (i.e. contains more than one element) having $0(1)$ has a maximal filter (ideal). Furthermore it can be shown that for distributive lattices each maximal filter is prime. By duality the result also holds for ideals. This can be strengthened for Boolean algebras to ∇ is maximal iff ∇ is prime.

Given a Boolean algebra **A** and a filter ∇ in **A** we can define an equivalence relation on **A** by $a \sim b$ iff $a \cup -b \in \nabla$ and $b \cup -a \in \nabla$, where $-$ denotes complementation. Then $\mathbf{A}/\nabla$, the set of equivalence classes under $\sim$, is a Boolean algebra and the mapping $\phi : \mathbf{A} \xrightarrow{\text{onto}} \mathbf{A}/\nabla$ given by $\phi(a) = [a]_{\sim}$ is called the natural homomorphism. From this we also have ∇ is maximal iff $\mathbf{A}/\nabla$ is the two-element Boolean algebra.

3.1.4. Remark. It is easy to see that each lattice isomorphic to a set lattice (lattice of subsets of a set where join and meet are union and intersection, respectively) is distributive since a set lattice is distributive. The next result due to A. Stone shows that the converse holds, i.e. each distributive lattice is isomorphic to a set lattice.

Let $P(\mathbf{A})$ denote the set of all prime filters of **A**, $h(a)$ the set of all filters $\nabla \in P(\mathbf{A})$ such that $a \in \nabla$, $\mathbf{P}(\mathbf{A})$ the class of all sets $h(a)$ with $a \in \mathbf{A}$.

3.1.5. Proposition. **A** a distributive lattice implies h is an isomorphism of **A** onto $\mathbf{P}(\mathbf{A})$.

Proof. We assume the following lemma which is proved by applying Zorn's lemma; for each $a,b \in \mathbf{A}$, a distributive lattice, $b \leqslant a$ does not hold implies there is some prime filter ∇ such that $a \notin \nabla$ and $b \in \nabla$.

If $a,b \in \mathbf{A}$, $a \neq b$, then one of $a \leqslant b$, $a \geqslant b$ does not hold. By the lemma there is a prime filter which belongs to exactly one of the sets $h(a)$, $h(b)$. Thus $h(a) \neq h(b)$ so h is 1-1.

If $\nabla \in h(a \cup b)$ then $a \cup b \in \nabla$. Since ∇ is prime either $a \in \nabla$ or $b \in \nabla$. Hence either $\nabla \in h(a)$ or $\nabla \in h(b)$, so

$\nabla \in h(a) \cup h(b)$, where $\cup$ denotes set union. Conversely if $\nabla \in h(a) \cup h(b)$ then either $a \in \nabla$ or $b \in \nabla$. Hence, since ($a \in \nabla$ and $a \leqslant b$ implies $b \in \nabla$) implies ($a \in \nabla$ and $b \in \mathbf{A}$ implies $a \cup b \in \nabla$), we have either $a \cup b \in \nabla$ or $a \cup b \in \nabla$, i.e. $a \cup b \in \nabla$ so $\nabla \in h(a \cup b)$.

$\nabla \in h(a \cap b)$ iff $a \cap b \in \nabla$ iff $a \in \nabla$ and $b \in \nabla$ iff $\nabla \in h(a)$ and $\nabla \in h(b)$ iff $\nabla \in h(a) \cap h(b)$, where $\cap$ is intersection. Hence h is a homomorphism.

Clearly h maps $\mathbf{A}$ onto $\mathbf{P}(\mathbf{A})$.

3.1.6. Remarks. h, $P(\mathbf{A})$, $\mathbf{P}(\mathbf{A})$ are called the Stone isomorphism, the Stone Space, the Stone lattice of $\mathbf{A}$, respectively. If we regard $\mathbf{P}(\mathbf{A})$ as a subbasis then it is easy to show that $P(\mathbf{A})$ is a topological space.

It can be shown that for a distributive lattice $\mathbf{A}$, $P(\mathbf{A})$ is a T_0-space, and if $\mathbf{A}$ has $\mathbf{1}$ then $P(\mathbf{A})$ is compact. This can be strengthened for Boolean algebras to $P(\mathbf{A})$ is T_2 and $\mathbf{P}(\mathbf{A})$ is the class of clopen sets of $P(\mathbf{A})$.

If we call a set lattice, which is closed with respect to set complementation, a field of sets then 3.1.5 shows that any Boolean algebra $\mathbf{A}$ is isomorphic to a field of sets, or more exactly $\mathbf{P}(\mathbf{A})$ is a field of subsets of $P(\mathbf{A})$ and h is an isomorphism from $\mathbf{A}$ onto $\mathbf{P}(\mathbf{A})$.

3.1.7. Remark. If $\mathbf{A}$ is a Boolean algebra, T an infinite set and $a = \cup^{\mathbf{A}}_{t\in T}\, a_t$ then $h(a) = \cup_{t\in T}\, h(a_t)$ (where $\cup^{\mathbf{A}}$ is join in $\mathbf{A}$, $\cup$ is union in $P(\mathbf{A})$) just in case T is finite. The one way is obvious, the other follows from the compactness of $P(\mathbf{A})$ and the 'clopenness' of $\mathbf{P}(\mathbf{A})$. Because each h(a) is closed in compact space $\mathbf{P}(\mathbf{A})$, h(a) is compact.

Hence, since each $h(a)$ is a member of the subbasis $\mathbf{P}(\mathbf{A})$ it is open; for each $h(a) = \cup_{t\in T} h(a_t)$ for some finite T by the compactness of $h(a)$. Hence $h(a) = h(\cup_{t\in T} a_t)$, i.e. $a = \cup^{\mathbf{A}}_{t\in T} a_t$ for some finite T.

Our next result shows that we can strengthen $h(a) \supset \cup_{t\in T} h(a_t)$ when T is infinite.

<u>3.1.8. Proposition.</u> Let $\mathbf{A}$ be a Boolean algebra, h the Stone isomorphism. If $a = \cup^{\mathbf{A}}_{t\in T} a_t$ then $h(a) - \cup_{t\in T} h(a_t)$ is a closed, nowhere dense subset of $P(\mathbf{A})$. By duality we have the corresponding statement for $\cap^{\mathbf{A}}$, $\cap$.

<u>Proof.</u> Let $H = h(a) - \cup_{t\in T} h(a_t)$. Then since each element of $\mathbf{P}(\mathbf{A})$ is clopen, H is closed.

Suppose H is not nowhere dense. Then H contains a non-empty open set and so there is an element $a_0 \in \mathbf{A}$ such that $h(a_0) \subset H$ and $a_0 \neq \mathbf{0}$. In view of 3.1.1, since $h(a_0) \subset h(a)$ then $a_0 \leqslant a$. So $a \neq a - a_0 \leqslant a$. Conversely $h(a\ ) \subset h(a) - h(a_t)$ for each $t \in T$. Hence $h(a_t) \subset h(a) - h(a_0) = h(a - a_0)$ for each $t \in T$. So in view of 3.1.1, $a_t \leqslant a - a_0$ for each $t \in T$. This contradicts $a = \cup^{\mathbf{A}}_{t\in T} a_t$.

<u>3.1.9. Definition.</u> Let $\mathbf{A}$ be a Boolean algebra and $\mathfrak{Q}$ be a set of infinite joins and meets in $\mathbf{A}$. A Boolean homomorphism from $\mathbf{A}$ into $\mathbf{B}$ is a $\mathfrak{Q}$-homomorphism $=_{df}$ it preserves all infinite joins and meets in $\mathfrak{Q}$.

<u>3.1.10. Remark.</u> Any isomorphism is necessarily a $\mathfrak{Q}$-homomorphism, so h regarded as a mapping from $\mathbf{A}$ into $\mathbf{P}(\mathbf{A})$ is a $\mathfrak{Q}$-homomorphism and in fact a $\mathfrak{Q}$-isomorphism. However, h regarded as a mapping from $\mathbf{A}$ into the field of all subsets of

$\mathcal{P}(\mathbf{A})$, $\mathcal{B}(\mathcal{P}(\mathbf{A}))$, whilst it is a monomorphism it is not a $\mathfrak{a}$-monomorphism except in some trivial cases.

It is important, then, to know under what condition does there exist a $\mathfrak{a}$-monomorphism h_0 from $\mathbf{A}$ into the field of all subsets of $\mathcal{P}(\mathbf{A})$. The next definition allows us to make such a condition.

<u>3.1.11. Definition.</u> Let $\mathbf{A}$ be a Boolean algebra. A maximal filter ∇ in $\mathbf{A}$ is a $\mathfrak{a}$-filter $=_{df}$ the natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\nabla$ is a $\mathfrak{a}$-homomorphism. We modify the notation of 3.1.4; $\mathcal{P}_{\mathfrak{a}}(\mathbf{A})$ is the set of $\mathfrak{a}$-filters in $\mathbf{A}$; $h_{\mathfrak{a}}$ and $\mathbf{P}_{\mathfrak{a}}(\mathbf{A})$ are defined similarly.

<u>3.1.12. Remark.</u> Equivalently a maximal and hence prime filter is a $\mathfrak{a}$-filter iff for each $s \in S'$, $a_s \in \nabla$ implies there is a $t \in T'_s$ with $a_{s,t} \in \nabla$, and for each $s \in S''$, $b_s \in \nabla$ implies there is a $t \in T''_s$ with $b_{s,t} \in \nabla$, where $\mathfrak{a}$ is the set $\{a_s = \cup^{\mathbf{A}}_{t \in T'_s} a_{s,t} : s \in S'\} \cup \{b_s = \cap^{\mathbf{A}}_{t \in T''_s} a_{s,t} : s \in S''\}$.

By definition $\mathcal{P}_{\mathfrak{a}}(\mathbf{A})$ is a subset of $\mathcal{P}(\mathbf{A})$ and $h_{\mathfrak{a}}(a) = h(a) \cap \mathcal{P}_{\mathfrak{a}}(\mathbf{A})$. Thus $h_{\mathfrak{a}}$ is a homomorphism from $\mathbf{A}$ into the field of all subsets of $\mathcal{P}_{\mathfrak{a}}(\mathbf{A})$, namely onto the field $\mathbf{P}_{\mathfrak{a}}(\mathbf{A})$.

It is easy to see that $h_{\mathfrak{a}}$ is a $\mathfrak{a}$-homomorphism from $\mathbf{A}$ into $\mathcal{B}(\mathcal{P}_{\mathfrak{a}}(\mathbf{A}))$.

<u>3.1.13. Proposition.</u> The following conditions are equivalent:

(i) every nonzero $a \in \mathbf{A}$ belongs to a $\mathfrak{a}$-filter;

(ii) $h_{\mathfrak{a}}$ is a $\mathfrak{a}$-monomorphism from $\mathbf{A}$ into $\mathcal{B}(\mathcal{P}_{\mathfrak{a}}(\mathbf{A}))$;

(iii) there is $\mathfrak{a}$-homomorphism h_0 from $\mathbf{A}$ into $\mathcal{B}(X)$ (the field of all subsets of a topological space X), for some X.

Proof. (i) implies $h_{\mathbf{0}}(a) \neq \phi$ for $a \neq \mathbf{0}$. It is easy to see that $h : \mathbf{A} \to \mathbf{B}$ is a monomorphism iff ($h(a) = \mathbf{0}_{\mathbf{B}}$ iff $a = \mathbf{0}_{\mathbf{A}}$), so this implies $h_{\mathfrak{a}}$ is a $\mathfrak{a}$-homomorphism in view of 3.1.12, i.e. (i) implies (ii).

(ii) implies (iii) trivially.

If (iii) holds and $a \neq \mathbf{0}$ then $h_{\mathfrak{a}}(a) \neq \phi$. Hence there is an $x_0 \in h_0(a)$. So since the set ∇ of all $b \in \mathbf{A}$ such that $x_0 \in h(b)$ is a $\mathfrak{a}$-filter and $a \in \nabla$ then (i) holds, i.e. (iii) implies (i).

3.1.14. Proposition. If the set

$$\mathfrak{a} = \{a_s = \cup^{\mathbf{A}}_{t\in T'_s} a_{s,t} : s \in S'\} \cup \{b_s = \cap^{\mathbf{A}}_{t\in T''_s} b_{s,t} : s \in S''\}$$

of infinite joins and meets is at most countable, then the set of all filters which are not $\mathfrak{a}$-filter is meagre in $P(\mathbf{A})$, and every non-zero element $a \in \mathbf{A}$ belongs to a $\mathfrak{a}$-filter.

Proof. This is due to H. Rasiowa and R. Sikorski in [13].

By the definition of $\mathfrak{a}$-filters, a maximal filter ∇ is not a $\mathfrak{a}$-filter iff it belongs to one of the sets $h(a_s) - \cup_{t\in T'_s} h(a_{s,t})$ for $s \in S'$, $\cap_{t\in T''_s} h(b_{s,t}) - h(b_s)$ for $s \in S''$, where h is the Stone isomorphism and $\cup$, $\cap$ as usual denote union, intersection.

In view of 3.1.8 each of these sets is nowhere dense. Hence the set $\mathbf{M}$ of all maximal filters which are not $\mathfrak{a}$-filters is meagre in $P(\mathbf{A})$ since $\mathfrak{a}$ is countable.

If $a \neq \mathfrak{a}_{\mathbf{A}}$, then $h(a)$ is open non-empty set of a compact T_2 space $P(\mathbf{A})$. It can be shown (the argument is similar to that of complete metric as T_2- allows us to form a decreasing sequence of closed sets which are separate from any meagre set and which by compactness is non-empty) that any compact T_2 space has the Baire Property. Thus $P(\mathbf{A})-\mathbf{M}$ is dense, so that $h(a)-\mathbf{M}$ is non-empty, i.e. there is a $\mathfrak{a}$-filter ∇ such that $a \in \nabla$.

<u>3.1.15. Remark.</u> $\mathbf{A}$ is a complete Boolean algebra (cBa) $=_{df}$ for each non-empty set S, $\cap^{\mathbf{A}}S$, $\cup^{\mathbf{A}}S$ exist.

If $\mathbf{A}$ is a cBa, $\mathfrak{a}$ is the set of all infinite joins and meets in $\mathbf{A}$, then $P_{\mathfrak{a}}(\mathbf{A})$ is the field $B(P_{\mathfrak{a}}(\mathbf{A}))$ of all subsets of $P_{\mathfrak{a}}(\mathbf{A})$. In fact, each one point set ∇ where $\nabla \in P_{\mathfrak{a}}(\mathbf{A})$ is then the image of $a_{\nabla} = \cap^{\mathbf{A}}\{a : a \in \nabla\}$. Hence for each $P \subset P_{\mathfrak{a}}(\mathbf{A})$ we have $P = h_{\mathfrak{a}}(a)$, where $a = \cup^{\mathbf{A}}\{a_{\nabla} : \nabla \in P\}$.

<u>3.1.16. Remark.</u> Each Boolean algebra $\mathbf{A}$ is isomorphic to a subalgebra of a cBa. For example the Stone isomorphism maps $\mathbf{A}$ into $B(P(\mathbf{A}))$ which is complete. However, in general this isomorphism does not preserve infinite joins and meets. We now give a monomorphism of $\mathbf{A}$ into a cBa which does preserve infinite joins and meets.

Let us say a set $A \subset X$, X a topological space, satisfies the Baire Condition iff there is an open set G with A-G, G-A meagre. It is easy to see that for a given topological space, the class of sets satisfying the Baire condition forms a field of subsets of X. Moreover this field contains all open subsets and closures of all open sets since $\mathrm{cl}A - A$ is nowhere dense for A open.

Consider the Stone Space $P(\mathbf{A})$ for Boolean algebra $\mathbf{A}$. Let $\mathbf{B}$ be the field of all sets $A \subset P(\mathbf{A})$ satisfying the Baire Condition. Let Δ be the ideal of all sets $A \subset P(\mathbf{A})$ which are meagre. The Boolean algebra $\mathbf{A}^* = \mathbf{B}/\Delta$ is called the minimal extension of $\mathbf{A}$. The mapping $h^* : \mathbf{A} \to \mathbf{A}^*$ defined by $h^*(a) = |h(a)|$ for $a \in \mathbf{A}$, where h is the Stone isomorphism of $\mathbf{A}$ onto $\mathbf{P}(\mathbf{A})$ and $|\ |$ denotes the equivalence classes in $\mathbf{B}/\Delta$, is a homomorphism, which can be shown to be 1-1.

h^* is called the canonical monomorphism of $\mathbf{A}$ into $\mathbf{A}^*$. We next show that $\mathbf{A}^*$ is a cBa and h^* preserves all infinite joins and meets. The proof of this and the fact that this embedding of $\mathbf{A}$ satisfies a certain nice universal condition is due to R. Sikorski in [13].

3.1.17. Proposition. $\mathbf{A}^*$ is a cBa and h^* preserves all infinite joins and meets.

Proof. For each $A \in \mathbf{B}$ there is an open set $G \in P(\mathbf{A})$ such that $|A| = |G|$ iff the Baire Condition holds for A.

Hence (1) each element of $\mathbf{A}^*$ is of the form $|G|$ for some open $G \in P(\mathbf{A})$. Obviously for any indexed set $\{G_t\}_{t\in T}$ of open subsets of $P(\mathbf{A})$,

$$|G_t| \leqslant |\cup_{t\in T} G_t| \quad \text{for each } t \in T.$$

If G open, $|G| \in \mathbf{A}^*$ is such that $|G_t| \leqslant |G|$ for each $t \in T$, then we can show $G_t \subset \mathrm{cl}G$ for each t using the Baire Condition and the properties of nowhere dense sets. Hence $\cup_{t\in T} G_t \subseteq \mathrm{cl}G$, so that $|\cup_{t\in T} G_t| \leqslant |\mathrm{cl}G| = |G| \cup |\mathrm{cl}G - G|$, and $|\mathrm{cl}G - G|$ is the zero of $\mathbf{A}^*$ since $\mathrm{cl}G - G$ is nowhere dense.

I.e. (2) $\cup^{\mathbf{A}^*}_{t\in T} |G_t| = |\cup_{t\in T} G_t|$ which exists.

(1) and (2) imply $\mathbf{A}^*$ is complete.

Using 3.1.8 we have $|h(a) - \cup_{t\in T} h(a_t)|$ is the zero of $\mathbf{A}^*$, so noting that $|h(a)| = |h(a) - \cup_{t\in T} h(a_t)| \cup^{\mathbf{A}^*} |\cup_{t\in T} h(a_t)|$

we have h^* preserves infinite joins. Using this and the infinite de Morgan Laws gives h^* preserves infinite meets.

3.2. TOPOLOGICAL BOOLEAN ALGEBRAS.

3.2.1. Definition. A topological Boolean algebra (tBa) $=_{df}$ a Boolean algebra $\mathbf{A}$ with an operation $\mathbf{I} : \mathbf{A} \to \mathbf{A}$ given by: for each $a,b \in \mathbf{A}$, (i) $\mathbf{I}(a \cap b) = \mathbf{I}a \cap \mathbf{I}b$ (ii) $\mathbf{I}a \leqslant a$ (iii) $\mathbf{I}\mathbf{I}a = \mathbf{I}a$ (iv) $\mathbf{I}\mathbf{1} = \mathbf{1}$. The operation $\mathbf{I}$ is called an interior operation.

3.2.2. Remark. If X is a topological space and $\mathbf{A}$ is a field of subsets of X such that $\mathbf{I}A \in \mathbf{A}$ for each $A \in \mathbf{A}$, where $\mathbf{I}$ is the interior operator int of topology, then $\mathbf{A}$ is a tBa which is called a topological field of subsets of X. In particular $\mathcal{B}(X)$, the field of all subsets of a topological space X, is a topological field. Moreover $\mathcal{B}(X)$ is a cBa.

Since (i) - (iv) are just the axioms of int, $\mathbf{I}$ has roughly the same properties as int. For instance we call $\mathbf{I}a$ the interior of a; a is open $=_{df}$ $\mathbf{I}a = a$; define an operator $\mathbf{C} =_{df} -\mathbf{I}-$ so that $\mathbf{C}a$ is the closure of a; a is closed $=_{df}$ $\mathbf{C}a = a$.

A topological subalgebra of a topological Boolean algebra is a Boolean subalgebra closed under $\mathbf{I}$.

3.2.3. Definition. A class $\mathbf{B}$ of open elements of a tBa $\mathbf{A}$ is a basis of $\mathbf{A}$ $=_{df}$ each open element of $\mathbf{A}$ is the join of some elements in $\mathbf{B}$.

A class $\mathbf{B}_0$ of open elements of a tBa $\mathbf{A}$ is a subbasis of $\mathbf{A}$ $=_{df}$ the class $\mathbf{B}$, comprising $\mathbf{0},\mathbf{1}$ and all finite meets of elements in $\mathbf{B}_0$, is a basis of $\mathbf{A}$.

3.2.4. Proposition. For every class $\mathbf{B}_0$ of elements of a cBa $\mathbf{A}$, there is exactly one interior operation $\mathbf{I}$ in $\mathbf{A}$ such that $\mathbf{B}_0$ is a subbasis of the tBa $\mathbf{A}$ with interior operation $\mathbf{I}$.

Proof.

(i) Existence. Let $\mathbf{B}$ be the class comprising $\mathbf{0},\mathbf{1}$ and all finite meets of elements in $\mathbf{B}_0$. For each $A \in \mathbf{A}$ let $\mathbf{I}A$ be the join of all elements $B \in \mathbf{B}$ such that $B \leqslant A$. This exists since A is a cBa. Properties (ii)-(iv) follow immediately from this definition of $\mathbf{I}$. Since for each $B_1, B_2 \in \mathbf{B}$, $B_1 \cap B_2 \in \mathbf{B}$, then

$$\begin{aligned}
\mathbf{I}A \cap \mathbf{I}A' &= \cup\{B \in \mathbf{B} : B \leqslant A\} \cap \cup\{B' \in \mathbf{B} : B' \leqslant A'\} \\
&= \cup\{B \in \mathbf{B} \text{ and } B' \in \mathbf{B} : B \leqslant A \text{ and } B' \leqslant A'\} \\
&\quad [\text{since cBa's have infinite distributive laws}] \\
&\leqslant \cup\{B \cap B' \in \mathbf{B} : B \cap B' \leqslant A \cap A'\} \\
&\leqslant \mathbf{I}(A \cap A').
\end{aligned}$$

Since $\cup_t(a_t \cap b_t) \leqslant \cup_t a_t \cap \cup_t b_t$ then $\mathbf{I}(A \cap A') \leqslant \mathbf{I}A \cap \mathbf{I}A'$, i.e. (i) holds. Therefore $\mathbf{B}$ is a basis of tBa $\mathbf{A}$ and so $\mathbf{B}_0$ is a subbasis.

(ii) Uniqueness. If $\mathbf{I}$ is an interior operation in $\mathbf{A}$ such that $\mathbf{B}_0$ is a subbasis of $\{\mathbf{A},\mathbf{I}\}$, then $\mathbf{B}$ defined above is a basis. Consequently the open element $\mathbf{I}A$ is the join of all elements $B \in \mathbf{B}$ such that $B \leqslant \mathbf{I}A$. But elements in $\mathbf{B}$ are open so that $B \in \mathbf{B}$ implies $B \leqslant \mathbf{I}A$ iff $B \leqslant A$. Thus $\mathbf{I}A$ is the join of all elements $B \in \mathbf{B}$ such that $B \leqslant A$. Hence $\mathbf{I}$ coincides with the $\mathbf{I}$ defined above.

3.2.5. Proposition. Let $\mathbf{B}_0$ be a subalgebra of a tBa $\mathbf{B}$. Every interior operation $\mathbf{I}_0$ in $\mathbf{B}_0$ can be extended to an interior operation $\mathbf{I}$ in $\mathbf{B}$ in such a way that open elements of $\mathbb{B}_0$ constitute a basis for $\mathbb{B}$.

Proof. This is due to J.C.C. McKinsey and A. Tarski.

By 3.2.4 there is an interior operation $\mathbb{I}$ in $\mathbb{B}$ such that the class $\mathbb{G}_0$ of all open elements in $\mathbb{B}_0$ is a subbasis of $\mathbb{B}$. Since the class $\mathbb{G}_0$ contains $\mathbb{0},\mathbb{1} \in \mathbb{B}$ and $b_1 \cap b_2 \in \mathbb{G}_0$ for each b_1, $b_2 \in \mathbb{G}_0$, then $\mathbb{G}_0$ is a basis of the tBa $\mathbb{B}$.

By definition $\mathbb{I}a$ is the join of all $b \in \mathbb{G}_0$ such that $b \leqslant a$. Conversely $\mathbb{I}_0 a$ is the greatest element $b \in \mathbb{G}_0$ such that $b \leqslant a$, since $b \leqslant a$ iff $b \leqslant \mathbb{I}a$ for b open. Hence $\mathbb{I}a = \mathbb{I}_0 a$.

3.2.6. Remark. A function h of a tBa $\mathbb{A}$ into a tBa $\mathbb{B}$ is a topological Boolean homomorphism $=_{df}$ it preserves the Boolean operations $\cap$, $\cup$, $-$ and the interior operation $\mathbb{I}$. We analogously define topological Boolean epi-, mono-, isomorphism.

It is obvious that a topological isomorphism regarded just as a topological map from $\mathbb{A}$ into $\mathbb{B}$ (i.e. regarding $\mathbb{A}$, $\mathbb{B}$ as topological spaces ignoring their Ba structure) is a homeomorphism. We now consider under what conditions will a mapping from topological space X into topological space Y induce a topological Boolean homomorphism.

3.2.7. Proposition. Given a function $\phi : X \to Y$, X,Y topological spaces, then the rule $h(B) = \phi^{-1}[B]$ for each $B \subset Y$ defines a Boolean homomorphism from $\mathcal{B}(Y)$ into $\mathcal{B}(X)$.

Proof.

Function $B = C \in \mathcal{B}(Y)$ implies $h(B) = \phi^{-1}[B] = \{x \in X : \phi(x) \in B\}$

$$= \{x \in X : \phi(x) \in C\}$$

$$= \phi^{-1}[C] = h(C).$$

Moreover if range $\phi \neq Y$ then for some $B, \phi^{-1}[B] = \phi$(the empty set) and since $\phi \in \mathcal{B}(X)$ then ϕ^{-1} maps $\mathcal{B}(Y)$ into $\mathcal{B}(X)$.

Homomorphism ϕ^{-1} preserves all the set theoretical operations which are the Boolean operations for $\mathcal{B}(Y)$, $\mathcal{B}(X)$.

3.2.8. Remarks. We now show some equivalent definitions for the topological notions of continuous and open functions which use the interior operator definition of topology rather than the open set definition.

If X,Y are topological spaces, then $\phi : X \to Y$ is continuous iff for each $B \subseteq Y$, $\phi^{-1}[\text{int } B] \subseteq \text{int } \phi^{-1}[B]$. ϕ continuous implies for each B open in Y, $\phi^{-1}[B]$ is open in X. In particular for each $B \subset Y$, int B is open in Y so $\phi^{-1}[\text{int } B]$ is open in X, i.e. $\phi^{-1}[\text{int } B] = \text{int } \phi^{-1}[\text{int } B]$. $\text{int } B \subseteq B$ implies $\phi^{-1}[\text{int } B] \subseteq \phi^{-1}[B]$ so that $\text{int } \phi^{-1}[\text{int } B] \subseteq \text{int } \phi^{-1}[B]$. Thus $\phi^{-1}[\text{int } B] \subseteq \text{int } \phi^{-1}[B]$. Suppose B is open in Y so that int B = B. By definition $\text{int } \phi^{-1}[B] \subseteq \phi^{-1}[B]$. By supposition $\phi^{-1}[B] = \phi^{-1}[\text{int } B] \subseteq \text{int } \phi^{-1}[B]$, i.e. $\phi^{-1}[B] = \text{int } \phi^{-1}[B]$ so that $\phi^{-1}[B]$ is open in X, i.e. ϕ is continuous.

If X and Y are topological spaces, then $\phi : X \to Y$ is open iff for each $A \subseteq Y$, $\phi(\text{int } A) \subseteq \text{int } \phi(A)$. ϕ open implies $\phi(\text{int } A) = \text{int } \phi(\text{int } A)$. $\text{int } A \subseteq A$ implies $\phi(\text{int } A) \subseteq \phi(A)$. Hence $\text{int } \phi(\text{int } A) \subseteq \text{int } \phi(A)$, so $\phi(\text{int } A) \subseteq \text{int } \phi(A)$. A open implies int A = A. By supposition $\phi(A) = \phi(\text{int } A) \subseteq \text{int } \phi(A)$. Hence since $\text{int } \phi(A) \subseteq \phi(A)$ we have $\phi(A)$ is open, i.e. ϕ is open.

3.2.9. Proposition. The Boolean homomorphism of 3.2.7 is topological iff ϕ is a continuous open function.

Proof. This is due to A. Wallace. As remarked before we can identify the topological interior operator int and $\mathbf{I}$, in view of their similar properties. We must show that $h(\mathbf{I}B) = \mathbf{I}h(B)$ for each $B \subseteq Y$.

By the first part of 3.2.8 we have ϕ is continuous iff $\phi^{-1}[\mathbf{I}B] \subseteq \mathbf{I}\phi^{-1}[B]$ for each $B \subseteq Y$. I.e. ϕ is continuous iff $h(\mathbf{I}B) \subseteq \mathbf{I}h(B)$ for each $B \subseteq Y$. From the second part of 3.2.8 it is easy to show ϕ is open iff $\mathbf{I}\phi^{-1}[B] \subseteq \phi^{-1}[\mathbf{I}B]$ for each $B \subseteq Y$, i.e. ϕ is open iff $\mathbf{I}h(B) \subseteq h(\mathbf{I}B)$ for each $B \subseteq Y$. Combining these two results gives the proposition.

3.2.10. Proposition. For each tBa $\mathbf{A}$ there exists a complete tBa (ctBa) $\mathbf{A}^*$ and a topological monomorphism h of $\mathbf{A}$ into $\mathbf{A}^*$ such that h preserves all infinite joins and meets in $\mathbf{A}$, and the elements h(a) where a is open in $\mathbf{A}$ are a basis for $\mathbf{A}^*$.

Proof. This is due to H. Rasiowa in [13].

Let h be the canonical Boolean monomorphism of $\mathbf{A}$ into its minimal extension $\mathbf{A}^*$ as in 3.1.16. h preserves all infinite meets and joins in $\mathbf{A}$. In 3.2.5 let $\mathbf{B}_0 = h(\mathbf{A})$ and $\mathbf{B} = \mathbf{A}^*$. The monomorphism h induces an interior operation in $h(\mathbf{A})$ via the interior operation $\mathbf{I}$ in $\mathbf{A}$ as follows: $\mathbf{I}h(a) =_{df} h(\mathbf{I}a)$ for $a \in \mathbf{A}$. This is well-defined since $h : \mathbf{A} \to h(\mathbf{A})$ is an isomorphism. By 3.2.5 we can extend this interior operation to $\mathbf{A}^*$, so that $\mathbf{A}^*$ is a ctBa and h is the required topological monomorphism.

3.2.11. Proposition. For each tBa $\mathbf{A}$ there exists a topological space X and a topological monomorphism h of

$\mathbf{A}$ into $\mathcal{B}(X)$. In addition we assume that the cardinal of X is $\leqslant 2^m$ where m is the cardinal of $\mathbf{A}$, and that $\{h(a): a \text{ is open in } \mathbf{A}\}$ is a basis for X. Moreover, given any countable set $\mathfrak{Q}$ of infinite joins and meets we can always assume that h is a $\mathfrak{Q}$-monomorphism of $\mathbf{A}$ into $\mathcal{B}(X)$.

Proof. The 1st part is due to J.C.C. McKinsey and A. Tarski, the 2nd part to H. Rasiowa and R. Sikorski.

By 3.1.13, 3.1.14 we may assume $X = \mathcal{P}_{\mathfrak{Q}}(\mathbf{A})$ and h is $h_{\mathfrak{Q}}$. Hence h is a Boolean $\mathfrak{Q}$-homomorphism of $\mathbf{A}$ onto a field $\mathbf{B}_0$ of subsets of X. We define an interior operation in $\mathbf{B}_0$ as in 3.2.10, i.e. $\mathbf{I}h(a) = h(\mathbf{I}a)$ for $a \in \mathbf{A}$.

By 3.2.5 we extend this to the cBa $\mathcal{B}(X)$. Hence X becomes a topological space with basis $\{h(a): a \text{ is open in } \mathbf{A}\}$.

The cardinal of X is estimated as follows: 2^m is the cardinal of the power set of $\mathbf{A}$. Since every prime filter is a subset of $\mathbf{A}$ the cardinal of the set of all maximal prime filters is $\leqslant 2^m$.

3.3. PSEUDO-BOOLEAN ALGEBRAS.

3.3.1. Remarks. The definition of $\cap$-complement (also known as the pseudo-complement) can be generalized. An element c is the pseudo-complement of a relative to b $=_{df}$ c is the greatest element such that $a \cap c \leqslant b$ (denoted by $a \Rightarrow b$). Equivalently, for each $x \in \mathbf{A}$, $x \leqslant (a \Rightarrow b)$ iff $a \cap x \leqslant b$.

A lattice $\mathbf{A}$ is said to be relatively pseudo-complemented $=_{df}$ $a \Rightarrow b$ exists for each $a,b \in \mathbf{A}$. It is easy to see that such a lattice has the unit element $\mathbf{1}$, but in general such lattices do not have the zero element $\mathbf{0}$.

A relatively pseudo-complemented lattice with zero element is called a pseudo-Boolean algebra (pBa).

If we define $a \Rightarrow b =_{df} -a \cup b$ then it is easy to see that each Ba is a pBa. We can show pBas are distributive lattices. Moreover just as for Bas, given filters ∇ or ideals Δ, we can define quotient pBas $\mathbf{A}/\nabla$, $\mathbf{A}/\Delta$, for pBa $\mathbf{A}$, and the natural homomorphism from $\mathbf{A}$ onto $\mathbf{A}/\nabla$, $\mathbf{A}/\Delta$. Also maximal filters in pBas are equivalent to filters which, for each $a \in \mathbf{A}$, contain exactly one of a, $a \Rightarrow \mathbf{0}$ ($=_{df} -a$, the pseudo-complement of a).

<u>3.3.2. Remark</u>. For any tBa $\mathbf{B}$, $G(\mathbf{B}) =_{df}$ the set of all open elements in $\mathbf{B}$. Since the join and meet of two open elements is open, $G(\mathbf{B})$ is a sublattice of $\mathbf{B}$. We next show that $G(\mathbf{B})$ is a pBa.

<u>3.3.3. Proposition</u>. The lattice $G(\mathbf{B})$ of all open elements in a tBa $\mathbf{B}$ is a pBa. Moreover if $\Rightarrow_{\mathbf{A}}$, $-_{\mathbf{A}}$ denote relative pseudo-complement and pseudo-complement in lattice $\mathbf{A}$, then $a \Rightarrow_{G(\mathbf{B})} b = \mathbf{I}(a \Rightarrow_{\mathbf{B}} b)$ (1) and $-_{G(\mathbf{B})} a = \mathbf{I}(-_{\mathbf{B}} a)$ (2) for each $a, b \in G(\mathbf{B})$.

<u>Proof</u>. To show $G(\mathbf{B})$ is a pBa we need to show for each $a, b \in G(\mathbf{B})$, $a \Rightarrow b$ exists, and that $G(\mathbf{B})$ has a zero element. We do this by establishing (1) and (2).

For each $a, b, x \in G(\mathbf{B})$, $a \cap x \leqslant b$ iff $x \leqslant (a \Rightarrow_{\mathbf{B}} b)$, since $\Rightarrow_{\mathbf{B}}$ is the relative complement in $\mathbf{B}$ and $G(\mathbf{B}) \subseteq \mathbf{B}$. Since x is open, $x \leqslant (a \Rightarrow_{\mathbf{B}} b)$ iff $x \leqslant \mathbf{I}(a \Rightarrow_{\mathbf{B}} b)$, i.e. $a \cap x \leqslant b$ iff $x \leqslant \mathbf{I}(a \Rightarrow_{\mathbf{B}} b)$. But for each $a, b, x \in G(\mathbf{B})$,

$a \cap x \leqslant b$ iff $x \leqslant a \Rightarrow_{G(\mathbf{B})} b$. Hence ① holds.

$-_{G(\mathbf{B})} a =_{df} a \Rightarrow_{G(\mathbf{B})} \mathbf{0}_{G(\mathbf{B})}$, and $\mathbf{0}_{G(\mathbf{B})}$ does exist and equals $\mathbf{0}_{\mathbf{B}}$. By ① we have $-_{G(\mathbf{B})} a = \mathbf{I}(a \Rightarrow_{\mathbf{B}} \mathbf{0}_{\mathbf{B}}) =_{df} \mathbf{I}(-_{\mathbf{B}} a)$.

3.3.4. Remark. If a and b are clopen in $\mathbf{B}$ then, since $\mathbf{B}$ is a Ba and so $a \Rightarrow_{\mathbf{B}} b = -_{\mathbf{B}} a \cup b$, $a \Rightarrow_{G(\mathbf{B})} b$ and $-_{G(\mathbf{B})}$ a are clopen in $\mathbf{B}$.

If $\mathbf{B} = \mathcal{B}(X)$ for some topological space X, we write $G(X)$ instead of $G(\mathcal{B}(X))$.

We define pseudo-Boolean homomorphism in the obvious way.

The next proposition is due to J.C.C. McKinsey and A. Tarski and gives a very exact relationship between pBas and tBas.

3.3.5. Proposition. For every pBa $\mathbf{A}$, there exists a tBa $\mathbf{B}$ such that $\mathbf{A} = G(\mathbf{B})$.

Proof. We first show the following lemma.

Lemma. For each distributive lattice $\mathbf{A}$ with the zero and unit element, there exists a Ba $\mathbf{B}$ such that

(i) $\mathbf{A}$ is a sublattice of $\mathbf{B}$ and $\mathbf{0}_{\mathbf{B}} = \mathbf{0}_{\mathbf{A}}$, $\mathbf{1}_{\mathbf{B}} = \mathbf{1}_{\mathbf{A}}$,

(ii) for each element $b \in \mathbf{B}$, $b = (a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n')$ for some $a_1, a_1', \ldots a_n, a_n' \in \mathbf{A}$.

Proof. We have shown in 3.1.5 that each distributive lattice is isomorphic to a set lattice, so it suffices to prove this lemma for the case when $\mathbf{A}$ is a lattice of subsets of a space X; i.e. join and meet are union and intersection, $\mathbf{0}_{\mathbf{A}} = \phi$, $\mathbf{1}_{\mathbf{A}} = X$.

Moreover given a Ba $\mathbf{A}$, a non-empty set $\mathbf{A}_0 \subseteq \mathbf{A}$, the Boolean subalgebra $\mathbf{A}'$ generated by $\mathbf{A}_0$ is the set of all elements of the form $\bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} a_{ij}$, where for each i,j either $a_{ij} \in \mathbf{A}_0$ or $-a_{ij} \in \mathbf{A}_0$. Further if $\mathbf{0}_{\mathbf{A}}, \mathbf{1}_{\mathbf{A}} \in \mathbf{A}_0$, then $\mathbf{A}'$ is the set of all elements of the form $\bigcap_{i=1}^{m}(-_{\mathbf{A}}a_i \cup b_i)$, where $a_i, b_i \in \mathbf{A}_0$. This holds since any $-a_1 \cup \ldots \cup -a_r \cup b_1 \cup \ldots \cup b_s = -a \cup b$, where
$a = a_1 \cap \ldots \cap a_r$ for $r \neq 0$, $a = \mathbf{1}_{\mathbf{A}}$ for $r = 0$, where $a_i \in \mathbf{A}_0$;
$b = b_1 \cup \ldots \cup b_s$ for $s \neq 0$, $b = \mathbf{0}_{\mathbf{A}}$ for $s = 0$, where $b_i \in \mathbf{A}_0$.

Hence this holds for $\mathcal{B}(X)$, for the X of the first paragraph of this proof. Since $a \Rightarrow_{\mathbf{A}} b =_{df} -_{\mathbf{A}} a \cup b$ for Ba $\mathbf{A}$, we have (i) and (ii) holding for the Ba generated by our set lattice $\mathbf{A}$.

As we have previously remarked, a pBa is a distributive lattice which has zero and unit elements. Given our pBa $\mathbf{A}$, let $\mathbf{B}$ be the Ba satisfying (i) and (ii) in the above lemma.

We define an interior operation on $\mathbf{B}$ as follows:
If $b \in \mathbf{B}$ is of the form $(a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n')$ then

$$\mathbf{I}b =_{df} (a_1 \Rightarrow_{\mathbf{A}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{A}} a_n')$$

<u>$\mathbf{I}$ well defined</u> $a \cap (a \Rightarrow_{\mathbf{A}} a') \leqslant a'$ implies $a \Rightarrow_{\mathbf{A}} a' \leqslant -_{\mathbf{B}} a \cup a' = a \Rightarrow_{\mathbf{B}} a'$, i.e. $a \Rightarrow_{\mathbf{A}} a' \leqslant a \Rightarrow_{\mathbf{B}} a'$ (*) and $a \cap (a \Rightarrow_{\mathbf{A}} a') \leqslant a'$ follows from, for each $x \in \mathbf{A}$, $a \cap x \leqslant b$ iff $x \leqslant (a \Rightarrow_{\mathbf{A}} b)$. Thus

$(a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n') \leqslant a \Rightarrow_{\mathbf{B}} a'$ iff
$(a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n') \cap a \leqslant a'$
implies by (*) $(a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n') \leqslant (a \Rightarrow_{\mathbf{B}} a')$
implies $(a_1 \Rightarrow_{\mathbf{A}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{A}} a_n') \cap a \leqslant a'$
implies $(a_1 \Rightarrow_{\mathbf{A}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{A}} a_n') \leqslant a \Rightarrow_{\mathbf{A}} a'$
[since for each $x \in \mathbf{A}$, $a \cap x \leqslant b$ iff $x \leqslant a \Rightarrow_{\mathbf{A}} b$].

Hence $(a_1 \Rightarrow_{\mathbf{B}} a_1') \cap \ldots \cap (a_n \Rightarrow_{\mathbf{B}} a_n') = (b_1 \Rightarrow_{\mathbf{B}} b_1') \cap \ldots \cap (b_m \Rightarrow_{\mathbf{B}} b_m')$

implies $(a_1 \Rightarrow_A a_1') \cap \ldots \cap (a_n \Rightarrow_A a_n') = (b_1 \Rightarrow_A b_1') \cap \ldots \cap (b_m \Rightarrow_A b_m')$, i.e. $\mathbf{I}$ is well-defined.

<u>$\mathbf{I}$ interior operation.</u> Property 3.2.1 (i) follows immediately from the definition of $\mathbf{I}$. Property 3.2.1 (ii) follows from $a \Rightarrow_A a' \leqslant a \Rightarrow_B a'$. Since obviously $\mathbf{I}b \in \mathbf{A}$ for each $b \in \mathbf{B}$ and $\mathbf{I}a = a$ for each $a \in \mathbf{A}$, we have property 3.2.1 (iii). $\mathbf{I}a = a$ for each $a \in \mathbf{A}$ implies property 3.2.1 (iv).

Moreover $\mathbf{I}b \in \mathbf{A}$ for each $b \in \mathbf{B}$ and $\mathbf{I}a = a$ for each $a \in \mathbf{A}$ imply $\mathbf{A} = G(\mathbf{B})$. The relative pseudo-complement and pseudo-complement in $\mathbf{A}$ coincide with those induced in $G(\mathbf{B})$ by the interior operation as in 3.3.3, since if they exist they are uniquely determined by the join and meet.

<u>3.3.6. Remark.</u> It is easy to prove the following condition also holds for 3.3.5. If $\mathbf{A}$ is finite, then $\mathbf{B}$ is finite. If the cardinal of $\mathbf{A}$ is $m \geqslant \aleph_0$ then the cardinal of $\mathbf{B}$ is m.

The next propositions are the analogues of 3.2.10, 3.2.11.

<u>3.3.7. Proposition.</u> For each pBa $\mathbf{A}$ there exists a cpBa $\mathbf{A}^*$ and a monomorphism of $\mathbf{A}$ into $\mathbf{A}^*$ preserving all infinite joins and meets in $\mathbf{A}$.

<u>Proof.</u> We assume the following lemma which is easily proved.

<u>Lemma.</u> $\mathbf{B}$ a tBa, $\mathbf{A} = G(\mathbf{B})$, $a_t \in \mathbf{A}$ for each $t \in T$, implies

(i) $\bigcup^{\mathbf{A}}_{t \in T} a_t$ exists iff $\bigcup^{\mathbf{B}}_{t \in T} a_t$ exists and then they are equal,

(ii) if $\bigcap^{\mathbf{B}}_{t \in T} a_t$ exists then $\bigcap^{\mathbf{A}}_{t \in T} a_t$ exists and $\bigcap^{\mathbf{A}}_{t \in T} a_t = \mathbf{I} \bigcap^{\mathbf{B}}_{t \in T} a_t$.

By 3.3.5 given pBa $\mathbf{A}$ we may assume $\mathbf{A} = G(\mathbf{B})$ for tBa $\mathbf{B}$. By 3.2.10 there exists a ctBa $\mathbf{B}^*$ and a topological monomorphism h of $\mathbf{B}$ into $\mathbf{B}^*$, preserving all infinite joins and meets in $\mathbf{B}$, and the elements h(a) for each $a \in \mathbf{B}$ form an open basis for $\mathbf{B}^*$.

By the lemma, since $\mathbf{B}^*$ is complete, $G(\mathbf{B}^*)$ is a complete lattice. Since in 3.3.5 $\mathbf{I}a = a$ for each $a \in \mathbf{A}$, $\mathbf{I}b \in \mathbf{A}$ for each $b \in \mathbf{A}$, then the restriction of a topological homomorphism h from Ba $\mathbf{B}$ into $\mathbf{B}'$, to $G(\mathbf{B})$, is a pseudo- Boolean homomorphism of $G(\mathbf{B})$ into $G(\mathbf{B}')$. Hence h restricted to $\mathbf{A}$ is a pseudo-Boolean monomorphism of $\mathbf{A}$ into $\mathbf{A}^* (= G(\mathbf{B}^*))$. Using the lemma we can show h preserves all infinite joins and meets in $\mathbf{A}$.

<u>3.3.8. Definition</u>. Let $\mathbf{A}$ be a pBa and

$\mathfrak{q} = \{a_s = \bigcup^{\mathbf{A}}_{t \in T_s} a_{s,t} : s \in S\}$ be a given set of infinite joins in $\mathbf{A}$. Let X be a topological space. A pseudo-Boolean homomorphism of $\mathbf{A}$ into $G(X)$ is a $(\mathfrak{q}, \cap)$-homomorphism $=_{df}$

(1) $h(a_s) = \bigcup^{G(X)}_{t \in T_s} h(a_{s,t})$ for each $s \in S$, and

(2) $h(b) = \mathbf{I}\bigcap^{G(X)}_{t \in T} h(b_t)$ for each indexed set $\{b_t : t \in T\}$ such that $b = \bigcap^{\mathbf{A}}_{t \in T} b_t$ exists. As usual $\bigcup^{G(X)}$, $\bigcap^{G(X)}$ denote union, intersection.

<u>3.3.9. Proposition</u>. For each pBa $\mathbf{A}$ there is a topological space X and a pseudo-Boolean homomorphism h of $\mathbf{A}$ into $G(X)$. In addition we can assume that the sets h(a), $a \in \mathbf{A}$, are a basis for X and that if the cardinal of $\mathbf{A}$ is $m \geqslant \aleph_0$ then the cardinal of X is $\leqslant 2^m$.

Moreover, given any countable set $\mathfrak{q}$ of infinite joins in $\mathbf{A}$ we can assume that h is a $(\mathfrak{q}, \cap)$-monomorphism of $\mathbf{A}$ into $G(X)$.

<u>Proof.</u> The first part is due to J.C.C. McKinsey and A. Tarski, the second to H. Rasiowa and R. Sikorski.

By 3.3.5 we can assume that $\mathbf{A} = G(\mathbf{B})$, where $\mathbf{B}$ is a tBa such that the cardinal of $\mathbf{B} \leqslant m$. By the lemma of 3.3.7 we have $a_s = \cup^{\mathbf{B}}_{t \in T_s} a_{s,t}$ for $s \in S$. By 3.2.11 there is a topological space X whose cardinal is $\leqslant 2^m$ and a topological monomorphism of $\mathbf{B}$ into $B(X)$. Moreover we can assume by 3.2.11 that $h(a_s) = \cup^{B(X)}_{t \in T_s} h(a_{s,t})$ for $s \in S$ - (1), and that the class of sets $h(a)$, $a \in \mathbf{A}$, is an open basis for X.

Also $b = \cap^{\mathbf{A}}_{t \in T} b_t$ implies $b \leqslant b_t$ for each $t \in T$

implies $h(b) \leqslant h(b_t)$ for each $t \in T$

implies $h(b) \leqslant \cap^{B(X)}_{t \in T} h(b_t)$.

$I \cap^{B(X)}_{t \in T} h(b_t)$ is open so it equals $\cup^{B(X)}_{s \in S} h(a_s')$. Since $\cup^{B(X)}_{s \in S} h(a_s') = I \cap^{B(X)}_{t \in T} h(b_t)$, $h(a_s') \leqslant h(b_t)$ for each t. Hence $a_s' \leqslant b_t$ for each t. This implies $a_s' \leqslant b$ so that $h(a_s') \leqslant h(b)$ for each s. I.e. $\cup^{B(X)}_{s \in S} h(a_s') \leqslant h(b)$.

Hence int $\cap^{B(X)}_{t \in T} h(b_t) \leqslant h(b) \leqslant \cap^{B(X)}_{t \in T} h(b_t)$ and $h(b)$ open implies if $b = \cap^{\mathbf{A}}_{t \in T} b_t$ then $h(b) = \cap^{B(X)}_{t \in T} h(b_t)$. - (2)

In view of the lemma of 3.3.7, (1) implies $h(a_s) = \cup^{G(X)}_{t \in T_s} h(a_{s,t})$ for $s \in S$ and (2) implies $h(b) = I \cap^{G(X)}_{t \in T} h(b_t)$. Then the monomorphism h restricted to $\mathbf{A}$ is a pseudo-Boolean $(\mathfrak{q}, \cap)$-monomorphism of $\mathbf{A}$ into $G(X)$ by the argument of 3.3.7.

<u>3.3.10. Remarks.</u> The next proposition, due to R. Sikorski, shows that a countable pBa with a countable set $\mathfrak{q}$ of infinite joins is $(\mathfrak{q}, \cap)$-isomorphic to a sublattice of $G(X_0)$, where X_0 is a set of irrational numbers. In the next section we shall show that intuitionistic logic can be regarded as a pBa. We mentioned in Chapter One that the set of irrationals was homeomorphic to Baire Space. Thus we have a further reason for looking at Baire Space when discussing intuitionistic second order arithmetic.

For this proposition we use the term separable space for a space which has a countable open basis (such a space is usually said to satisfy the 2nd axiom of countability). Also we need the result from topology that for each space which has cardinal $\leqslant 2^{\aleph_0}$, there is a set X_0 of irrational numbers and an open continuous function ϕ of X_0 onto X.

<u>3.3.11. Proposition.</u> For each countable pBa $\mathbf{A}$ and for every given ennumerable set $\mathfrak{q}$ of infinite joins in $\mathbf{A}$, there is a set X_0 of irrational numbers and a pseudo-Boolean $(\mathfrak{q}, \cap)$-monomorphism of $\mathbf{A}$ into $G(X_0)$.

<u>Proof.</u> By 3.3.9 there is a separable space X with cardinal $\leqslant 2^{\aleph_0}$ and a $(\mathfrak{q}, \cap)$-monomorphism h of $\mathbf{A}$ into $G(X)$. By 3.3.10 there is a set X_0 of irrational numbers and an open continuous function ϕ of X_0 onto X.

By 3.2.9 and applying the argument of 3.3.7 about the restriction of topological homomorphisms from Ba $\mathbf{B}$ into Ba $\mathbf{B}'$, to $G(\mathbf{B})$, this ϕ induces a pseudo-Boolean homomorphism of $G(X)$ into $G(X_0)$, namely $h_1(A) = \phi^{-1}[A]$, $A \in G(X)$.

Moreover since ϕ is onto, h_1 is 1-1.

Define h_0 from **A** into $G(X_0)$ by $h_0(a) =_{df} h_1(h(a))$ for $a \in$ **A**. Since h is a $(\mathfrak{q}, \cap)$-monomorphism, $h_1(a_s) = h_1(\cup^{B(X)}_{t \in T_s} h_0(a_{s,t}))$ for $s \in S$ and $h_0(b) = h_1(\mathbf{I}\cap^{B(X)}_{t \in T} h(b_s))$. In view of the lemma of 3.3.7 we have $h_0(a_s) = h_1(\cup^{G(X)}_{t \in T_s} h(a_{s,t}))$ for $s \in S$ and $h_0(b) = h_1(\mathbf{I}\mathbf{I}\cap^{G(X)}_{t \in T} h(b_t)) = h_1(\mathbf{I}\cap^{G(X)}_{t \in T} h(b_t))$.

Since joins and meets in $G(X)$ are unions and intersections which h_1 preserves, and since by 3.2.9 h_1 is topological, i.e. preserves $\mathbf{I}$, $h_0(a_s) = \cup^{G(X)}_{t \in T_s} h_1(h(a_{s,t}))$ for $s \in S$ and $h_0(b) = \mathbf{I}\cap^{G(X)}_{t \in T} h_1(h(b_t))$. In view of the lemma of 3.3.7 applied to cBas, h_0 is a pseudo-Boolean $(\mathfrak{q}, \cap)$-monomorphism of **A** into $G(X_0)$.

3.4. INTUITIONISTIC FIRST ORDER FORMALIZED THEORIES.

3.4.1. Remarks. We firstly give the familiar syntax and syntactical rules which enable us to discuss first order formalized theories.

We shall deal with a fixed formalized language $L = \{A,T,F\}$ of the first order. A is the alphabet of the first order, namely the ordered system $\{V, \Xi, \{\Phi_m\}_{m \in N}, \{P_m\}_{m \in N}, L_1, L_2, Q, U\}$ such that each member of this system is disjoint, V, Ξ are infinite, $\cup_{m \in N} P_m$, $L_1 \cup L_2$ are non empty and Q has two elements. Elements in V are free individual variables (lower case latin), elements in Ξ are bound individual variables (early lower case greek), elements in Φ_m are m-argument functors with Φ_0 also called individual constants (later lower case greek), elements in P_m are m-argument predicates, elements in L_1 are unary propositional connectives, elements in L_2 are binary propositional connectives, elements in Q are quantifiers, elements in U the auxiliary signs.

This so far is a general description of the alphabet. For our purposes we have $L_1 = \{\neg\}$, $L_2 = \{\vee, \wedge, \rightarrow\}$, $Q = \{\exists, \forall\}$ and for later work in Chapter Four and Five, $P_2 = \{=\}$, $\Phi_0 = \{0\}$, $\Phi_1 = \{+\}$, $\Phi_2 = \{+, \times\}$, U is $\{(,)\}$.

T is the set of terms: the least set of finite sequences of signs formed from signs in A such that:

(i) each free individual and each individual constant is in T;

(ii) if τ is a m-argument functor with free individual variables $x_1, \ldots, x_m$ and if $\tau_1, \ldots, \tau_m$ are in T, then the result of substituting $\tau_1, \ldots, \tau_m$ for $x_1, \ldots, x_m$ in τ, is in T. A term of type (i) is called elementary.

F is the set of all formulas: the least set of finite sequences of signs in A such that:

(i) if A is $P(x_1, \ldots, x_m)$ for m-argument predicate P and free individual variables $x_1, \ldots, x_m$, $\tau_1, \ldots, \tau_m$ are terms, then the result of substituting $\tau_1, \ldots, \tau_m$ for $x_1, \ldots, x_m$ in A, is in F;

(ii) if A is in F then $\neg A$ is in F;

(iii) if A and B are in F then $A \vee B$, $A \wedge B$, $A \rightarrow B$ are in F;

(iv) if $A(x)$ is in F then for each bound variable ξ which does not appear in $A(x)$, $\exists\xi A(\xi)$, $\forall\xi A(\xi)$ are in F. A formula of type (i) is called elementary.

In A it is usual to call $\{V, L_1, L_2, U\}$ the logical symbols, and $\{\{\Phi_m\}, \{P_m\}, Q\}$ the parameters. Elementary formulas are what are usually called atomic formulas.

Although it is not essential that the set of all signs in L is countable (then L is called countable) we will assume this condition so that we can use 3.3.11.

A finite sequence of formulas $A_1, \ldots, A_n$ is said to be a formal proof of a formula A in L from a set S of formulas provided that A_n is A and for each $j \leqslant n$ either A_j is one of the logical axioms of L, or is a formula in S, or is an immediate consequence of some formulas A_{j_1}, A_{j_2}, j_1, $j_2 < j$ by modus ponens, or one of the following rules of inference.

(i) $A(x) \to B$ implies $\exists\xi A(\xi) \to B$ where B contains no occurrence of x and ξ is free in $A(x)$;

(ii) $A \to B(x)$ implies $A \to \forall\xi B(\xi)$ where A contains no occurrence of x and ξ is free in $B(x)$.

The logical axioms of L are the set of intuitionistic tautalogies

$T_1((A \to B) \to ((B \to C) \to (A \to C)))$ $T_2(A \to (A \vee B))$ $T_3(B \to (A \vee B))$

$T_4((A \to C) \to ((B \to C) \to ((A \vee B) \to C)))$ $T_5((A \wedge B) \to A)$

$T_6((A \wedge B) \to B)$ $T_7((C \to A) \to ((C \to B) \to (C \to (A \wedge B))))$

$T_8((A \to (B \to C)) \to ((A \wedge B) \to C))$ $T_9(((A \wedge B) \to C) \to (A \to (B \to C)))$

$T_{10}((A \wedge \neg A) \to B)$ $T_{11}((A \to (A \wedge \neg A)) \to A)$, and the two schemas $A(\tau|x) \to \exists\xi A(\xi)$ $\forall\xi A(\xi) \to A(\tau|x)$, where $A(\tau|x)$ is the substituition of τ for x in $A(x)$, so that in view of the distinction between bound and free individual variables $A(\tau|x)$ is the $S_x^\tau(A|x))$ of 2.1.5.

The intuitionistic consequence operation C_i is defined as: given a set of logical axiom schemes A_i and the rules of inference, for each set of formulas S in L, $C_i(S)$ is the set of all formulas A in L for which there is a formal proof of A from S in L by means of A_i and the rules of inference (in which case A is said to be derivable in L from S).

$\{L, C_i\}$ is called the intuitionistic predicate calculus based on L. A formula A is derivable in $\{L, C_i\} =_{df}$ A is in $C_i(\phi)$. A formalized elementary intuitionistic theory $\zeta = \{L, C_i, A\}$ is a system where A is some set of formulas (the set of axioms for theory ζ). Formulas in $C_i(A)$ are theorems of theory ζ. By convention $\{L, C_i, \phi\}$ is identified with $\{L, C_i\}$.

3.4.2. Definition. Given a non-empty set $\mathbf{A}$, an m-argument operation in $\mathbf{A} =_{df}$ a function $o: \mathbf{A}^m \to \mathbf{A}$. An (abstract) algebra $=_{df} \{\mathbf{A}, \{o_\phi\}_{\phi \in \Phi}\}$. A generalized operation in $\mathbf{A} =_{df}$ a function $0 : \mathcal{D} \to \mathbf{A}$, where $\mathcal{D}$ is a class of subsets of $\mathbf{A}$. The $S \in \mathcal{D}$ are called admissable sets for generalized operation 0. A generalized algebra $=_{df}$ $\{\mathbf{A}, \{o_\phi\}_{\phi \in \Phi}, \{0_\psi\}_{\psi \in \Psi}\}$. If $\mathcal{D}$ is the power set of $\mathbf{A}$ then $\{\mathbf{A}, \{o_\phi\}, \{0_\psi\}\}$ is a complete generalized algebra.

3.4.3. Remarks. In view of the definition of F in 3.4.1, we see that for any intuitionistic theory $\zeta = \{L, C_i, A\}$, $\{F, \{\vee, \wedge, \to, \neg\}\}$ is an algebra (of formulas of language L).

Define a relation $\approx$ on F as follows: $A \approx B =_{df}$ both $A \to B$ and $B \to A$ are theorems in ζ. It is easily shown via the intuitionistic tautologies that $\approx$ is an equivalence relation in F and preserves all operations $\vee, \wedge, \to, \neg$ (i.e. it is a congruence relation). Hence we can form the quotient algebra $F/\!\approx\; = \{\{\|A\| : A \in F\}, \{\vee_\approx, \wedge_\approx, \to_\approx, \neg_\approx\}\}$, where

$\|A\|$ is the equivalence class of A under $\approx$. This is called the algebra of the theory ζ denoted by $\mathcal{U}(\zeta)$. This idea is due to Lindebaum.

3.4.4. Proposition. The algebra $\mathcal{U}(\zeta)$ of any intuitionistic theory ζ is a pBa. Moreover $\|A\| \leqslant_{\approx} \|B\|$ iff $(A \rightarrow B)$ is a theorem of ζ, $\|A\| = 1$ iff A is a theorem of ζ, $\|A\| \neq \phi$ iff A is irrefutable in ζ, $\mathcal{U}(\zeta)$ is non-degenerate iff ζ is consistent. Furthermore for each $B(x)$,
$\|\exists\xi B(\xi)\| = \cup_{\tau \in T} \|B(\tau)\|$, $\|\forall\xi B(\xi)\| = \cap_{\tau \in T}\|B(\tau)\|$ where $\cup$, $\cap$ are the infinite join and meet in pBa.

Proof. Define $A \leqslant B =_{df} (A \rightarrow B)$ is a theorem in ζ. If any instances of $T_0 = (A \rightarrow A)$ and T_1 are theorems in ζ then $\leqslant$ induces an ordering on $F/\approx$ given by: $\|A\| \leqslant_{\approx} \|B\|$ iff $(A \rightarrow B)$ is a theorem in ζ (1).

If $T_0 - T_7$ are theorems in ζ then $\leqslant_{\approx}$ is a lattice ordering, i.e. $\mathcal{U}(\zeta)$ is a lattice. For example by (1), T_2, T_3, $\|A\| \leqslant_{\approx} \|(A \vee B)\|$, $\|B\| \leqslant_{\approx} \|(A \vee B)\|$. Suppose $\|C\|$ in $\mathcal{U}(\zeta)$ is such that $\|A\| \leqslant_{\approx} \|C\|$, $\|B\| \leqslant_{\approx} \|C\|$, i.e. $(A \rightarrow C)$, $(B \rightarrow C)$ are theorems in ζ. By modus ponens on $(A \rightarrow C)$ and T_4 we have $((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$ is a theorem in ζ. By modus ponens on $(B \rightarrow C)$ and $((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$ we have $((A \vee B) \rightarrow C)$ is a theorem in ζ, i.e. $\|(A \vee B)\| \leqslant_{\approx} \|C\|$,
i.e. $\|(A \vee B)\| = \|A\| \cup \|B\|$, where $\cup$ is the join induced by $\leqslant_{\approx}$.

If $T_1 - T_9$ are theorems in ζ, then we can show T_0 is a theorem in ζ, hence by the above if $T_1 - T_9$ are theorems in ζ then $\mathcal{U}(\zeta)$ is a lattice. Moreover for each A,B, $\|A\| \Rightarrow \|B\| = \|(A \rightarrow B)\|$ (*) where $\Rightarrow$ denotes the relative pseudo-complement induced by $\leqslant_{\approx}$. The proof is similar to that used to show $\mathcal{U}(\zeta)$ is a lattice. Furthermore if ⊛ holds it is easy to see that $\|A\| = \mathbf{1}$ iff A is a theorem in ζ. Hence $\mathcal{U}(\zeta)$ is a relatively pseudo-complemented lattice.

If $T_1 - T_{10}$ are theorems in ζ then $\mathcal{U}(\zeta)$ is a pBa. This follows since $T_1 - T_9$ imply $\mathcal{U}(\zeta)$ is a relatively pseudo-complemented lattice and T_{10} implies $\|(A \wedge \neg A)\|$ is the zero of $U(\zeta)$. Moreover if T_{11} is a theorem of ζ we can show that $-\|A\| = \|\neg A\|$, where $-$ is the pseudo-complement induced by $\leqslant_{\approx}$. The proof is similar to showing $\mathcal{U}(\zeta)$ is a lattice.

Furthermore if $T_1 - T_9$ hold then the lattice $\mathcal{U}(\zeta)$ has an element different from $\mathbf{1}$ iff there is a formula which is not a theorem. Hence $\mathcal{U}(\zeta)$ is not degenerate iff ζ is consistent.

Therefore since $T_1 - T_{11}$ are theorems in any intuitionistic theory ζ we have $\mathcal{U}(\zeta)$ is a pBa where $\cup$, $\cap$, $\Rightarrow$, $-$ correspond to $\vee_{\approx}$, $\wedge_{\approx}$, $\rightarrow_{\approx}$, $\neg_{\approx}$.

To show the last part we prove the following lemma.

<u>Lemma</u>. If T_0, T_1 are theorems in ζ and T' is any set of terms containing infinitely many free variables then for each B(x),

$\|\exists\xi B(\xi)\| = \mathrm{lub}_{\tau \in T'} \|B(\tau)\|$, $\|\forall\xi B(\xi)\| = \mathrm{glb}_{\tau \in T'} \|B(\tau)\|$, in the ordered set $\mathcal{U}(\zeta)$.

<u>Proof</u>. Firstly we note we have the following derived rule of inference: if $\exists\xi A(\xi) \rightarrow B$ then $A(\tau|x) \rightarrow B$. For suppose $\exists\xi A(\xi) \rightarrow B \in \zeta$ and $A(\tau|x) \in \zeta$, then $\exists\xi A(\xi) \in \zeta$ by the axiom schema $A(\tau|x) \rightarrow \exists\xi A(\xi)$ and modus ponens. Hence $B \in \zeta$ by $\exists\xi A(\xi) \rightarrow B \in \zeta$ and modus ponens. I.e. $A(\tau|x) \rightarrow B \in \zeta$.

$\exists\xi B(\xi) \to \exists\xi B(\xi)$ is of the form T_0 so is a theorem of ζ. Applying the above rule of inference gives $B(\tau) \to \exists\xi B(\xi)$ is a theorem of ζ, i.e. $\|B(\tau)\| \leqslant_{\approx} \|\exists\xi B(\xi)\|$ for each $\tau \in T'$.

Suppose there is an $\|A\| \in \mathcal{U}(\zeta)$ with $\|B(\tau)\| \leqslant \|A\|$ for each $\tau \in T'$. Since A contains a finite number of free variables and T' contains infinitely many free variables, then there is an individual variable $x \in T'$ such that x does not occur free in A. Thus $\|B(x)\| \leqslant \|A\|$, i.e. $B(x) \to A$ is a theorem. Hence using the rule of inference (i) in 3.4.1, $\exists\xi B(\xi) \to A$ is a theorem, i.e. $\|\exists\xi B(\xi)\| \leqslant \|A\|$.

Hence $\|\exists\xi B(\xi)\| = \mathrm{lub}_{\tau \in T'} \|B(\tau)\|$. Similarly $\|\forall\xi B(\xi)\| = \mathrm{glb}_{\tau \in T'} \|B(\tau)\|$.

We have seen that if $T_1 - T_9$ are theorems in ζ then T_0 holds and $\mathcal{U}(\zeta)$ is a lattice. Let $T' = T$, the set of all terms, and since $\bigcup_{\tau \in T} a_t$ for each T is unique and equal to $\mathrm{lub}_{\tau \in T}\, a_t$, etc, we have the desired result.

3.4.5. Remark. Hence $\mathcal{U}(\zeta)$ is a generalized algebra $\{\{\|A\| : A \in F\}, \{\cup, \cap, \Rightarrow, -\}, \{\cup, \cap\}\}$, the domain of the generalized operations being $\{\{\|B(\tau)\|\}_{\tau \in T} : B(x) \in F\}$. This is called the $\mathfrak{a}$-theory of ζ.

3.4.6. Remarks. A realization of a formalized language $L = \{A, T, F\}$ in a set $J \neq \phi$ and in a complete algebra $\{\mathbf{A}, \{\cup, \cap, \Rightarrow, -\}, \{\cup, \cap\}\} =_{df}$ a function R defined on the set of functors and on the set of predicates in L such that:

(a) R assigns to each m-argument functor ϕ in L an m-argument function ϕ_R in J, i.e. $\phi_R : J^m \to J$;

(b) R assigns to every m-argument predicate P in L an m-argument function $P_R : J^m \to \mathbf{A}$.

Every formula A in L can be considered as a function $A_R : J^{V_A} \to \mathbf{A}$, where V_A is the finite set of all free individual variables occuring in A. For this purpose we interpret:

(i) all free or bound individual variables occuring in A as ranging over J;

(ii) each functor ϕ in A as a function $\phi_R : J^m \to J$;

(iii) each predicate P in A as a function $P_R : J^m \to \mathbf{A}$;

(iv) $\vee, \wedge, \to, \neg$ as the signs of the corresponding operations $\cup, \cap, \Rightarrow, -$ in $\mathbf{A}$;

(v) the quantifiers as signs of the corresponding operations $\cup_{\xi \in J}$, $\cap_{\xi \in J}$ in $\mathbf{A}$.

By extension we can consider each formula A in L uniquely determining a function $A_R : J^V \to \mathbf{A}$, where V is as before the set of free individual variables in L. Indeed the inductive definition of A_R is as follows:
Let v be a valuation in J, i.e. $v : V \to J$. Then

$P(\tau_1, \ldots, \tau_k)_R(v) =_{df} P_R(\tau_{1R}(v), \ldots, \tau_{kR}(v))$, where $\tau_R(v)$ is defined inductively by:

$x_R(v) =_{df} v(x)$, $\phi(\tau_1, \ldots, \tau_m)_R(v) =_{df} \phi_R(\tau_{1R}(v), \ldots, \tau_{mR}(v))$,

$(B \vee C)_R(v) =_{df} B_R(v) \cup C_R(v)$, etc.

$\exists \xi B(x_0 | \xi)_R(v) =_{df} \cup_{j \in J} B_R(w_j)$, where $w_j(x) = v(x)$ iff $x \neq x_0$,
$w_j(x) = j$ iff $x = x_0$,

etc.

<u>3.4.7. Remark.</u> From now on we assume that $\{\mathbf{A}, \{\cup, \cap, \Rightarrow, -\}, \{\cup, \cap\}\}$ is a non-degenerate pBa. In view of 3.3.7 and the fact that completeness in an algebra and completeness in a lattice are the same, we can drop the condition that $\mathbf{A}$ is complete: i.e. any realization in J and in an incomplete pBa $\mathbf{A}$ can be considered as a realization in J and the complete pBa $\mathbf{A}^*$. However if only elements in

A are values of $A_R(v)$, and only infinite operations in **A** are used to determine $A_R(v)$, then there is no reason to introduce the extension **A*** of **A**; it is more natural to consider R as a realization in **A**.

The idea of interpreting a formula A of intuitionistic predicate calculus as a function A_R, where R is a realization in a non empty set J and a cpBa, is due to A. Mostowski.

3.4.8. Definitions. Given a fixed formalized language L and a fixed realization R of L in a set $J \neq \phi$ and a pBa **A**, a function $v : V \to J$ is a valuation of L in J as remarked before. A valuation v satisfies formula A in realization R $=_{df}$ $A_R(v) = 1$. Formula A is satisfiable in R $=_{df}$ there is a valuation v in R which satisfies A. Formula A is valid in R $=_{df}$ each valuation in R satisfies A. In this case we say R is a model for A. A set of formulas S in L is valid in R $=_{df}$ each $A \in S$ is valid in R. In this case we say R is a model of the intuitionistic theory $\zeta = \{L, C_i, A\}$; in short R is a model for A.

A realization or model of L is countable $=_{df}$ J is countable.

Formula A in L is intuitionistically valid $=_{df}$ A is valid in each realization R in each pBa **A**.

A topological realization of $L =_{df}$ a realization of L in a set $J \neq \phi$ and the pBa $G(X)$ of all open subsets of a topological space X. A model of an intuitionistic theory $\zeta = \{L, C_i, A\}$ in a set $J \neq \phi$ and in $G(X)$ is a topological model for ζ.

A semantic realization of $L =_{df}$ any realization of L in a set $J \neq \phi$ and in the two element Ba $\{0, 1\}$. A model of an intuitionistic theory $\zeta = \{L, C_i, A\}$ in a set $J \neq \phi$ and in the two element Ba $\{\mathbf{0}, \mathbf{1}\}$ is a semantic model for ζ.

3.4.9. Remarks. Note that if A is closed then V_A is empty so A_R is the constant function. Hence if A is satisfiable in R then $A_R(v) = \mathbf{1}$ for some v, and so since A_R is a constant function, $A_R(v) = \mathbf{1}$ for each v, i.e. A is valid in R. The converse holds, so that for closed formulas satisfiability and validity coincide.

Obviously each realization is a model for $\{L, C_i, \phi\}$, i.e. $\{L, C_i\}$.

The idea of semantic realization can be shown to be equivalent to the usual idea of structure as in H.B. Enderton [3]. Hence this idea of realization and the resultant definitions of satisfiablity, validity and model, due to H. Rasiowa, is more general than the usual idea of structure.

3.4.10. Remarks. In 3.4.8 the only restriction on J was that it had to be non-empty. We now want to show that any intuitionistic theory ζ has a topological model, so we look for a J which will do this. The natural J to take is T, the set of terms.

A canonical realization of terms $=_{df}$ a realization R in the set of terms T and a generalized algebra $\mathbf{A}$, such that to each m-argument functor ϕ, R assigns the corresponding m-argument operation ϕ in T, i.e. $\phi_R(\tau_1, \ldots, \tau_m) = \phi(\tau_1, \ldots, \tau_m)$.

3.4.11. Remark. It has been shown by R. Sikorski in [13] that if h is a $\mathbf{Q}$-homomorphism for $U(\zeta)$ into a non-degenerate pBa $\mathbf{A}$ and R^0 is a function defined on the set of all functors in L and on the set of all predicates in L such that:

(a) R^0 restricted to the set of all functors in L is the canonical realization of terms in L;

(b) for each m-argument predicate P in L, for each term $\tau_1, \ldots, \tau_m$ in L; $P_{R^0}(\tau_1, \ldots, \tau_m) = h(\|P(\tau_1 \ldots \tau_m)\|)$;

then R^0 is the canonical realization for L determined by h.

<u>3.4.12. Remarks</u>. For every consistent theory $\zeta = \{L, C_i, A\}$ we have defined the $\mathfrak{Q}$-algebra $U(\zeta)$ which was shown to be a pBa. Because the set of formulas in L is countable and so the set of infinite joins and meets corresponding to the quantifiers is countable, by 3.3.11 we have a set X_0 of irrational numbers and a $\mathfrak{Q}$-monomorphism h of $U(\zeta)$ into $G(X_0)$. Hence the canonical realization R^0 for ζ determined by h, in the set of terms T and in pBa $G(X_0)$, is a topological model.

Moreover since $\|A\| = 1$ iff A is a theorem of ζ, by 3.4.4, and $h(\|A\|) = 1$ iff $\|A\| = 1$, because h is a monomorphism, and since A is a theorem iff any substitution of A is a theorem we have A is a theorem of ζ iff $A_{R^0}(v) = 1$ for each v, i.e. iff A is valid in R^0.

<u>3.4.13. Remark</u>. Suppose the intuitionistic theory $\zeta = \{L, C_i, A\}$ has the sign of equality $\mathfrak{e}$; that is the following axiom schemas are in A.

e_1	$\mathfrak{e}(x,x)$	x fixed free individual variable.
e_2	$\mathfrak{e}(x,y) \to \mathfrak{e}(y,x)$	x,y fixed distinct free individual variables.
e_3	$\mathfrak{e}(x,y) \to (\mathfrak{e}(y,z) \to \mathfrak{e}(x,z))$	x,y,z fixed distinct free individual variables.

e_4 $\mathcal{E}(x_1,y_1) \to (\mathcal{E}(x_2,y_2) \to (\ldots(\mathcal{E}(x_m,y_m)$
$\to \mathcal{E}(\phi(x_1,\ldots,x_m),\ \phi(y_1,\ldots,y_m)))\ldots))$
$x_1,\ldots,y_m$ fixed distinct free individual variables,
ϕ m-argument functor.

e_5 $\mathcal{E}(x_1,y_1) \to (\mathcal{E}(x_2,y_2) \to (\ldots(\mathcal{E}(x_m,y_m)$
$\to (P(x_1,\ldots,x_m) \to P(y_1,\ldots,y_m)))\ldots))$
$x_1,\ldots,y_m$ fixed free individual variables, P m-argument predicate.

A model R for ζ in $J \neq \phi$ and pBa **A** is an ordinary realization of $\mathcal{E} =_{df} \mathcal{E}_R(j_1, j_2) = 1$ iff $j_1 = j_2$.

<u>3.4.14. Proposition</u>. If an intuitionistic theory ζ with sign of equality $\mathcal{E}$ has a model R in a pBa **A** and a set $J \neq \phi$, then ζ has an ordinary model R' in **A** and J/*, where * is defined by: for each $j_1, j_2 \in J$, $j_1 * j_2 =_{df} \mathcal{E}_R(j_1, j_2) = 1$.

<u>Proof</u>. By $e_1 - e_3$ * is an equivalence relation. By e_4, e_5 we have $\phi_{R'}(|j_1|_*,\ldots,|j_m|_*) = |\phi_R(j_1,\ldots,j_m)|_*$ –①

$P_{R'}(|j_1|_*,\ldots,|j_m|_*) = P_R(j_1,\ldots,j_m)$ –②

and these define a realization R' of L.

Since $f : j \to |j|_*$ maps J onto J/* and ① and ② hold, it can be shown that $A_{R'}(fv) = A_R(v)$ for each $v \in J^V$. Hence R' is a model for ζ.

By ② and the definition of *, $\mathcal{E}_{R'}(|j_1|, |j_2|) =$ iff $\mathcal{E}_R(j_1, j_2) = 1$, i.e. iff $|j_1| = |j_2|$.

<u>3.4.15. Remark</u>. In view of 3.4.13, 3.4.14, every consistent theory with sign of equality has a topological model R' which is an ordinary realization of $\mathcal{E}$, and in which A is a theorem of ζ iff A is valid in R': much more than is used in the next chapter.

4. A TOPOLOGICAL MODEL OF INTUITIONISTIC SECOND ORDER ARITHMETIC

4.0.1. Remark. In the first section of this chapter we shall develop the idea of topological models from that of direct-power models. In the third section, by choosing our power set to be N^N and restricting the assignment of number variables to constant functions, we shall give a model of intuitionistic second order arithmetic.

4.1. GENERAL TOPOLOGICAL MODELS.

4.1.1. Remark. Recall that a structure $\mathcal{D}$ for first order arithmetic consists of a non empty set D called the domain, an element $0 \in D$, a function $^+$ from D into D and functions $+$, $\times$ from $D \times D$ into D, i.e. $\mathcal{D} = (D, 0, {}^+, +, \times)$.

The assignments $A \in \mathcal{A}$ are functions from the set of variables of the language into D. We write $A_{\mathbf{x}}^{x}$ for the assignment B such that $B(\mathbf{x}) = x$, $B(\mathbf{y}) = A(\mathbf{y})$ for each $\mathbf{y}$ not equal to $\mathbf{x}$.

We extend assignments to unique functions from the set of terms into D in the obvious way, i.e. for each $A \in \mathcal{A}$, $[\![\mathbf{0}]\!]_A = 0$, $[\![\mathbf{x}]\!]_A = A(\mathbf{x})$, $[\![\mathbf{a}^+]\!]_A = [\![\mathbf{a}]\!]_A^+$, etc.

For each $A \in \mathcal{A}$, satisfaction $\models [A]$ is the unique predicate on the set of formulas given in the obvious way, i.e. $\models \mathbf{a} = \mathbf{b}\,[A]$ iff $[\![\mathbf{a}]\!]_A = [\![\mathbf{b}]\!]_A$, $\models \neg\mathbf{A}[A]$ iff not $\models \mathbf{A}[A]$,

$\models \mathbf{A} \wedge \mathbf{B}\,[A]$ iff $\models \mathbf{A}\,[A]$ and $\models \mathbf{B}\,[A]$,

$\models \forall\mathbf{x}\, \mathbf{A}\,[A]$ iff $\models \mathbf{A}\,[A_{\mathbf{x}}^{x}]$ for each $x \in D$ etc.

If $\models \mathbf{A}\,[A]$ for each assignment A, then we say $\mathcal{D}$ is a model for $\mathbf{A}$.

<u>4.1.2. Remarks</u>. The structures in 4.1.1 are two valued in the sense that each formula is either satisfied or not satisfied by each assignment. We next introduce direct power structures where a formula has a value in a power set and that value is not necessarily the whole set or the empty set.

Given a structure $\mathcal{D} = (D, 0, {}^+, +, \times)$ and a non empty set I, the direct power structure $\mathcal{D}^I = (D^I, 0^I, {}^{+I}, +_I, \times_I)$ is defined as follows. The domain D^I is the class of functions from I into D. 0^I is the constant function in D^I that sends each $i \in I$ to 0. ${}^{+I}$ is the function from D^I into D^I defined pointwise by $f^{+I}(i) = f(i)^+$ for each $i \in I$, $f \in D^I$. $+_I$, $\times_I$ are the functions from $D^I \times D^I$ into D^I defined pointwise by $(f +_I g)(i) = f(i) + g(i)$, $(f \times_I g)(i) = f(i) \times g(i)$, for each $i \in I$, $f, g \in D^I$.

The assignments $A \in \mathcal{A}$ are functions from the set of variables into D^I. $A^f_{\mathbf{x}}$ is defined analogously to $A^x_{\mathbf{x}}$. Given $A \in \mathcal{A}$ we define an assignment $A(i)$ from the set of variables onto D by:

$A(i)(\mathbf{x}) = A(\mathbf{x})(i)$ for each $i \in I$, each $\mathbf{x}$.

For each $A \in \mathcal{A}$, $[\![\ ]\!]_A$ is the unique function from the set of terms into D^I given by $[\![\mathbf{0}]\!]_A = 0^I$, $[\![\mathbf{x}]\!]_A = A(\mathbf{x})$ for each $\mathbf{x}$, $[\![\mathbf{a}^+]\!]_A = [\![\mathbf{a}]\!]_A^{+I}$, for each term $\mathbf{a}$, etc.

For each $A \in \mathcal{A}$ we define the valuation $[\![\ ]\!]_A$, the unique function from the set of formulas into $\mathcal{P}(I)$ as follows:

$[\![\mathbf{a} = \mathbf{b}]\!]_A = \{i \in I\colon [\![\mathbf{a}]\!]_A(i) = [\![\mathbf{b}]\!]_A(i)\}$, $[\![\neg \mathbf{A}]\!]_A = -[\![\mathbf{A}]\!]_A$,

$[\![\mathbf{A} \wedge \mathbf{B}]\!]_A = [\![\mathbf{A}]\!]_A \cap [\![\mathbf{B}]\!]_A$ $[\![\forall \mathbf{x}\, \mathbf{A}]\!]_A = \cap\{[\![\mathbf{A}]\!]_{A^f_{\mathbf{x}}} : f \in D^I\}$, etc.

If $\llbracket \mathbf{A} \rrbracket_A = I$ then we say $\mathbf{A}$ is satisfied by A. If $\llbracket \mathbf{A} \rrbracket_A = I$ for each A, then we say $\mathcal{D}^I$ is a model of $\mathbf{A}$.

4.1.3. Remark. The important result which shows the connection between satisfaction and valuation is, for each i, each A, each $\mathbf{A}$, $i \in \llbracket \mathbf{A} \rrbracket_A$ iff $\models \mathbf{A}\,[A(i)]$ (*)

Firstly we show by induction on the complexity of terms that for each i, each A, each term $\mathbf{a}$, $\llbracket \mathbf{a} \rrbracket_A(i) = \llbracket \mathbf{a} \rrbracket_{A(i)}$. For example, for the induction basis, for each variable $\mathbf{x}$, $\llbracket \mathbf{x} \rrbracket_A(i) = A(\mathbf{x})(i) = A(i)(\mathbf{x}) = \llbracket \mathbf{x} \rrbracket_{A(i)}$.

From this result, by induction on the complexity of formulas we can show (*). For example, for the induction basis for $\mathbf{a} = \mathbf{b}$, $i \in \llbracket \mathbf{a} = \mathbf{b} \rrbracket_A$ iff $\llbracket \mathbf{a} \rrbracket_A(i) = \llbracket \mathbf{b} \rrbracket_A(i)$

iff $\llbracket \mathbf{a} \rrbracket_{A(i)} = \llbracket \mathbf{b} \rrbracket_{A(i)}$ by above

iff $\models \mathbf{a} = \mathbf{b}\,[A(i)]$.

Obviously when I is a singleton set $\mathcal{D}^I$ is isomorphic to $\mathcal{D}$.

4.1.4. Remarks. The idea of substructure is introduced in the obvious way. $\mathcal{D}'$ is a substructure of $\mathcal{D} =_{df} D' \subseteq D$ and $0' = 0$, and ${}^{+}{}'$, $+'$, $\times'$ are the restrictions of ${}^{+}$, $+$, $\times$ to D'.

Given a structure $\mathcal{D}$ and a non-empty set I, a substructure $\mathcal{L} = (L^I, 0^I, {}^{+}I, +_I, \times_I)$ is a subdirect power structure $=_{df}$ $\{f(i) : f \in L^I\} = D$ for each $i \in I$. In a subdirect power structure the assignments are from the variables into L^I so that the valuations $\llbracket \forall \mathbf{x}\, \mathbf{A} \rrbracket_A$, $\llbracket \exists \mathbf{x}\, \mathbf{A} \rrbracket_A$ are redefined by allowing the f in $A^f_{\mathbf{x}}$ to range over L^I instead of D^I.

The largest subdirect power structure of D is D^I

itself, and the smallest is when L^I is just the class of constant functions, i.e. $f(i) = x$ for some $x \in D$, for each i. This is isomorphic to D.

The condition $\{f(i) : f \in L^I\} = D$ for each i ensures that $i \in [\![A]\!]_A$ iff $\models A[A(i)]$ holds for subdirect power structures.

<u>4.1.5. Remarks</u>. We next define continuous subdirect power structures. Given a structure $\mathcal{D}$, a non-empty set I with topology $\mathcal{T}$, and a subdirect power structure $\mathcal{L}$, $\mathcal{L}$ is continuous $=_{df}$ each $f \in L^I$ is a continuous function from I into D, where D has the discrete topology.

The importance of this definition is that for each assignment A from the variables into L^I and for all terms $\mathbf{a}$, $\mathbf{b}$, $\{i \in I : [\![\mathbf{a}]\!]_A(i) = [\![\mathbf{b}]\!]_A(i)\}$ is an open set of $(I, \mathcal{T})$. Hence if we define $\mathcal{T}$-valuation $[\![\,]\!]_A^{\mathcal{T}}$ as follows:
$[\![\mathbf{a} = \mathbf{b}]\!]_A^{\mathcal{T}} = \{i : [\![\mathbf{a}]\!]_A(i) = [\![\mathbf{b}]\!]_A(i)\}$, $[\![\neg A]\!]_A^{\mathcal{T}} = \mathrm{int}(-[\![A]\!]_A^{\mathcal{T}})$,
$[\![A \to B]\!]_A^{\mathcal{T}} = \mathrm{int}((-[\![A]\!]_A^{\mathcal{T}}) \cup [\![B]\!]_A^{\mathcal{T}})$, $[\![A \wedge B]\!]_A^{\mathcal{T}} = [\![A]\!]_A^{\mathcal{T}} \cap [\![B]\!]_A^{\mathcal{T}}$,
$[\![A \vee B]\!]_A^{\mathcal{T}} = [\![A]\!]_A^{\mathcal{T}} \cup [\![B]\!]_A^{\mathcal{T}}$, $[\![\forall x\, A]\!]_A^{\mathcal{T}} = \mathrm{int} \cap \{[\![A]\!]_{A_x^f}^{\mathcal{T}} : f \in L^I\}$,
$[\![\exists x\, A]\!]_A = \cup\{[\![A]\!]_{A_x^f} : f \in L^I\}$; then the $\mathcal{T}$-valuations are always open sets of $(I, \mathcal{T})$. Moreover, since the only predicate we are dealing with is equality, we see that the continuous subdirect structure assigns to equality what the topological realization of Chapter Three assigns, namely open subsets of topological spaces. Therefore we shall call continuous subdirect structures topological structures, in short.

The largest topological structure for $\mathcal{D}$, topology $\mathcal{T}$, is the one for which L^I is the set of all continuous functions.

The set L^I is closed under the operations of D^I. For example for $+_I$, for each $r \in N$,

$$
\begin{aligned}
\{i \in I : (x +_I y)(i) = r\} &= \{i \in I : x(i) + y(i) = r\} \\
&= \{i \in I : x(i) = u \text{ and } y(i) = v \\
&\qquad \text{and } u + v = r\} \\
&= \cup\{\{i \in I : x(i) = u \\
&\qquad \text{and } y(i) = v\} : u + v = r\} \\
&= \cup\{\{i \in I : x(i) = u\} \\
&\qquad \cap\{i \in I : y(i) = v\} : u + v = r\},
\end{aligned}
$$

which is open. Hence $x +_I y$ is a continuous function from I into D.

4.1.6. Remarks. So far we have only considered first order arithmetic. Next we look at second order arithmetic, for which we have to introduce a domain of functions to our structure to which the function variables are assigned. Even though it is well known that choosing the domain of functions to be the class of all functions from D into D gives rise to incompleteness results and other problems, our structures will contain that choice of domain.

Hence a second order structure is $\mathcal{D} = (D, D^D, 0^+, +, \times)$, where quantifiers over functions are interpreted in the obvious way.

A second order direct power structure is $\mathcal{D}^I = (D^I, (D^D)^I, \dot{0}^I, {}^+I, +_I, \times_I)$. The assignments $A \in \mathcal{A}$ are extended to the set of function variables. Given $A \in \mathcal{A}$ we define an assignment $A(i)$ from the set of function variables into D by $A(i)(\mathfrak{f}) = A(\mathfrak{f})(i)$, for each i, each $\mathfrak{f}$.

For each $A \in \mathcal{A}$, $[\![\;]\!]_A$ is the unique function from the set of terms into D^I as before, with the addition
$[\![\mathbf{f}(\mathbf{a})]\!]_A(i) =_{df} [\![\mathbf{f}]\!]_A(i)[[\![\mathbf{a}]\!]_A(i)]$ - ⊛.

The valuations are extended by the valuations of the quantifiers over function variables in the obvious way. The condition ⊛ ensures that as before we have
$i \in [\![\mathbf{A}]\!]_A$ iff $\models \mathbf{A}[A(i)]$.

For subdirect power structures we make the obvious modifications. For continuous subdirect power structures, as well as each $f \in L^I$ being continuous, we require each $F \in (L^L)^I$ to be continuous. This means L^L must have a topology placed on it. The obvious topology is the induced product topology on D with the discrete topology, and then L^L has the induced subspace topology.

Such a definition ensures that $\{i \in I : [\![\mathbf{a}]\!]_A(i) = [\![\mathbf{b}]\!]_A(i)\}$ for all terms $\mathbf{a}$, $\mathbf{b}$, are open subsets in $(I, \mathcal{T})$. For example, to show $\{i \in I : [\![\mathbf{a}]\!]_A(i) = [\![\mathbf{b}]\!]_A(i)\}$ is open we write it as $\cup\{\{i \in I : [\![\mathbf{a}]\!]_A(i) = r\} \cap \{i \in I : [\![\mathbf{b}]\!]_A(i) = r\} : r \in L\}$ and check if $\{i \in I : [\![\mathbf{a}]\!]_A(i) = r\}$, $\{i \in I : [\![\mathbf{b}]\!]_A(i) = r\}$ are open. The most difficult case is when $\mathbf{a}$ is $\mathbf{f}(\mathbf{b})$.

However $\{i \in I : [\![\mathbf{f}(\mathbf{b})]\!]_A(i) = r\} =_{df}$

$$\{i \in I : [\![\mathbf{f}]\!]_A(i)[[\![\mathbf{b}]\!]_A(i)] = r\}$$

$$= \cup\{\{i \in I : [\![\mathbf{b}]\!]_A(i) = u \text{ and } [\![\mathbf{f}]\!]_A(i)[u] = r\} : u \in L\}$$

$$= \cup\{\{i \in I : [\![\mathbf{b}]\!]_A(i) = u \text{ and } [\![\mathbf{f}]\!]_A(i) \in \{h \in L^L : h(u) = r\}\} : u \in L\}$$

$$= \cup\{\{i \in I : [\![\mathbf{b}]\!]_a(i) = u\}$$

$$\cap \{i \in I : [\![\mathbf{f}]\!]_A(i) \in \{h \in L^L : h(u) = r\}\} : u \in L\}.$$

Here $\{i \in I : [\![\mathbf{b}]\!]_A(i) = u\}$ is open since $[\![\mathbf{b}]\!]_A$ is continuous in L^I;

$\{i \in I : [\![f]\!]_A(i) \in \{h \in L^L : h(u) = r\}\}$ is open since $[\![f]\!]_A$ is continuous in $(L^L)^I$ and $\{h \in L^L : h(u) = r\}$ is open in L^L by the induced subspace product topology.

Hence the valuations extended to the function variable quantifiers are open sets in $(I, \mathcal{T})$, so that $[\![\ ]\!]_A^{\mathcal{T}}$ can again be defined.

4.2. THE SYNTAX OF AN INTUITIONISTIC SECOND ORDER ARITHMETIC

4.2.1. Remark. In this section we shall give the syntax of an intuitionistic second order arithmetic which expresses formally the ideas, or slight modifications of them, of Chapter Two. It is this theory that is satisfied by the model of section three.

4.2.2. Definition. The formal symbols of the theory are: the logical symbols: parentheses (,) ; commas , ; propositional connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$; number variables x , y , z ,...; function (sequence) variables α, β, γ,...;
the parameters: quantifiers $\exists$, $\forall$; predicate symbol $=$; function symbols f_0 , f_1 , f_2 ,...,where f_0 is 0, f_1 is $^+$, f_2 is $+$, f_3 is $\times$; operators (Church's) λ;

4.2.3. Remark. There are two points to notice. Firstly instead of having an 'equality' symbol in the logical symbols we have an 'equality' predicate. This will be discussed later. Secondly the function symbols will be specified as needed. They are to be regarded as 'primitive recursive' functions of a specified number of variables and sequence variables.

<u>4.2.3. Definition</u>. We define functors and the set of terms T inductively as follows: The function variables are functors; if f_i is a 'primitive recursive' function of one number variable and no sequence variables then f_i is a functor; if x is a number variable and s is a term then $\lambda x(s)$ is a functor; the number variables are terms; if $t_1, \ldots, t_{k_i}$ are terms and $u_1, \ldots, u_{l_1}$ are functors then $f_i(t_1, \ldots, t_{k_i}, u_1, \ldots, u_{l_1})$ is a term (hence if i is 0, 0 is a term); if u is a functor and t is a term then $(u)(t)$ is a term.

<u>4.2.4. Definition</u>. We define the set of formulas F inductively as follows: If s, t are terms then $s = t$ is a formula; if A, B are formulas then $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$ are formulas; if x is a number variable and A is a formula then $\exists x(A)$, $\forall x(A)$ are formulas; if α is a function variable and A is a formula then $\exists\alpha(A)$, $\forall\alpha(A)$ are formulas.

<u>4.2.5. Remarks</u>. For simplicity, parentheses are omitted under the usual conventions. For ease of reading they may be changed to braces or brackets.

Free and bound occurrences of variables in terms, functors and formulas are distinguished in the usual way. However the λ operator λ also binds variables, so that we need an analogous definition for when term b is substitutable for x in s, where s contains λ. Similarly the result $S^b_x(A)$ of substituting term b for number variable x in A is extended in the obvious way to substituting functor u for function variable α in A.

4.2.6. Remark. We now give the axiom schemas for this theory of second order arithmetic:

Axiom schemas for the intuitionistic propositional calculus: just as in 2.1.

Axiom schemas for the intuitionistic second order predicate calculus:

1N If $C \to Ax$ then $C \to \forall x\, Ax$ 2N $\forall x\, Ax \to S_x^t(Ax)$

3N $S_x^t(Ax) \to \exists x\, Ax$ 4N If $Ax \to C$ then $\exists x\, Ax \to C$

where Ax is a formula, C is a formula which does not contain x free;

1F If $C \to A\alpha$ then $C \to \forall\alpha A\alpha$ 2F $\forall\alpha A\alpha \to S_\alpha^u(A\alpha)$

3F $S_\alpha^u(A\alpha) \to \exists\alpha A\alpha$ 4F If $A\alpha \to C$ then $\exists\alpha A\alpha \to C$

where $A\alpha$ is a formula, C is a formula which does not contain α free.

Axiom schemas for the intuitionistic number theory:

PIn $S_x^0(Ax) \to (\forall x(Ax \to S_x^{x^+}(Ax)) \to \forall x\, Ax)$

S1 $\forall x(\neg x^+ = 0)$ S2 $\forall x \forall y(x = y \to x^+ = y^+)$

S3 $\forall x \forall y\ (x^+ = y^+ \to x = y)$ =1 $\forall x \forall y \forall z(x = y \to (x = z \to y = z))$

A1 $\forall x((x + 0) = x)$ A2 $\forall x \forall y((x + y^+) = (x + y)^+)$

M1 $\forall x((x \times 0) = 0)$ M2 $\forall x \forall y((x \times y^+) = ((x \times y) + x))$

Axiom schemas for functions:

λ1 $\{\lambda x.r(x)\}(t) = S_\lambda^t(r(x))$ if t is substitutable for x in $r(x)$, for all terms $r(x)$ and t

=F $\forall x \forall y \forall\alpha(x = y \to \alpha(x) = \alpha(y))$

AC $\forall x \exists\alpha A(x,\alpha) \to \exists\alpha\forall x A(x, \lambda y.\alpha(2^x.3^y))$, where x is free for α in $A(x,\alpha)$.

Axiom schemas for certain 'primitive recursive' functions: These include the 'defining relations', such things as the exponential function of AC,

i.e. E1 $\forall x(x^0 = 1)$ E2 $\forall x \forall y(S_y^{y^+}(x^y) = (x^y \times x))$,
the predecessor function, the finite sum $\Sigma_{y<s}\ t(y)$, the finite product $\Pi_{y<s}\ t(y)$, the bounded least (μ) operator μ, etc., as well as versions of BI and Brouwers principle for numbers, namely

$$\mathbf{BI!}\ \ \{[\forall \alpha \exists! x R(\bar{\alpha} x) \wedge \forall w(Seq(w) \rightarrow (R(w) \rightarrow (A(w))) \wedge \forall w(Seq(w) \rightarrow (\forall s A(w*2^{s+1}) \rightarrow Aw))] \rightarrow A(1)\}$$

$$\mathbf{BC\text{-}N!}\ \ [\forall \alpha \exists! y A(\alpha, y) \rightarrow \exists \tau \forall \alpha \exists y \{\tau(\bar{\alpha}(y)) > 0 \wedge \forall x[\tau(\bar{\alpha}(x)) > 0 \rightarrow y = x] \wedge A(\alpha, \tau(\bar{\alpha}(y)) \dot{-} 1)\}]$$

4.2.7. Remarks. The formation rules provide = as a primitive symbol only between terms. When u, v are functors, $u = v$ will be an abbreviation for $\forall x(u(x) = v(x))$, where x is not free in u, v.

As we remarked before = is not a logical symbol. There are no axiom schemas which directly tell us how it is to be interpreted. However from the schema =1 we can derive the first three schemas of 3.4.13 [qv [9]], namely reflexivity, symmetry and transitivity. Moreover, the schemas **S2**, **=F**, tell us separately how it is to be interpreted on these functors $^+$, α. Indeed it can be shown by induction that all the 'primitive recursive' functions we introduce satisfy the fourth schema of 3.4.13 [qv [9], [10]] . Applying schema $\lambda 1$ to these 'primitive recursive' function schemas, will then show that all functors satisfy the fourth schema of 3.4.13. As we do not have predicates in our theory, we need not consider the fifth schema of 3.4.13.

4.2.8. Remarks. As we saw in Chapter One and Chapter Two, an essential technique for intuitionistic second order arithmetic is the assigning of finite sequences to numbers. In Chapter One we gave such an assignment. However, although this is a simple assignment, when we come to interpret our formal expression in the model, it gives rise to a complicated expression in the model. Instead we choose an equivalent assignment which is based on Gödel numbering.

Firstly we define what a prime number is, i.e.
$\vdash Pr(a) \leftrightarrow (a > 1 \wedge \neg\exists c(1 < c < a \wedge c \mid a))$, and indeed we have
$\vdash \neg Pr(0)$, $\vdash Pr(2)$, $\vdash Pr(3)$, $\vdash \neg Pr(4)$, etc.
Next we give a recursive equation which enumerates the primes:
$P_0 = 2$, $P_{i^+} = \mu b_{b < P_{i!+2}} [P_i < b \wedge Pr(b)]$ ⊛, where $!$ is the factorial function with recursive equation
$0! = 1$, $a^+! = a! \times a^+$. From ⊛ it is easy to show that
$\vdash Pr(a) \leftrightarrow \exists i(a = P_i)$ [qv [10]]. Next we give a definition of what informally is the power of the ith prime for a number whose prime decomposition is $P_0^{k_0} \times \ldots \times P_j^{k_j}$.
$(a)_i =_{df} \mu x_{x < a} [P_i^x \mid a \wedge \neg P_i^{x^+} \mid a]$. From this we can derive the two expected results, namely,
$\vdash (P_i^h)_i = h$, $\vdash a > 0 \leftrightarrow a = \Pi_{i < a} P_i^{(a)_i}$. Now we give the length function which informally is the number of factors in the prime decomposition of a number.

$$lh(a) =_{df} \Sigma_{i < a}\, sg((a)_i), \text{ where } sg(0) = 0,\ sg(a^+) = 1.$$

We can now define what is informally a sequence number,
$\vdash Seq(a) \leftrightarrow (a > 0 \wedge (\forall i < lh(a))((a_i) > 0))$. We can prove
[qv [10]] $\vdash Seq(a) \leftrightarrow a = \Pi_{i < lh(a)} P_i^{(a)_i}$,

i.e. a number is a sequence number iff its decomposition into prime factors is of the form $P_0^{k_0} \times \ldots \times P_i^{k_i}$, where the $k_i > 0$. If we define a concanentation function $*$ as follows, $a * b =_{df} a \times \Pi_{i < lh(a)} P_{lh(a)+i}^{(b)_i}$, then we can show [qv [10]] $\vdash a * 1 = a$, $\vdash (Seq(a) \wedge Seq(b) \rightarrow ((lh(a * b) = lh(a) + lh(b) \wedge Seq(a * b))$. Finally we give two functions which informally are functions from sequences into finite sequences.

$\bar{\alpha}(x) =_{df} \Pi_{i < x} P_i^{\alpha(i)+1}$, $\tilde{\alpha}(x) =_{df} \Pi_{i < x} P_i^{\alpha(i)}$. Indeed we can show [qv.[10]] that $\vdash (lh(\bar{\alpha}(x)) = x \wedge Seq(\bar{\alpha}x))$ and $\vdash Seq(a) \leftrightarrow \exists\alpha\exists x(a = \bar{\alpha}x)$, so that $^-$ is a function from sequences onto finite sequences.

To relate these ideas to those in Chapter One we give this informal account, noting first that our assignment, although more convenient to use in the formal language and model, is no longer onto, so that we have to add the antecedent $Seq(u)$ to the relevant clauses of **BI!**, **BC - N!**. Given numbers $a_0, \ldots, a_{t-1}$ which can be regarded as the first t values $\alpha(0), \ldots, \alpha(t-1)$ of a sequence α (the remaining values as yet undetermined), the finite sequence $\{a_0, \ldots, a_{t-1}\}$ is associated with the number $P_0^{a_0+1} \ldots P_{t-1}^{a_{t-1}+1}$ which we know is a sequence number. In the formal language we abbreviate $P_0^{a} P_1^{a_1} \ldots P_{t-1}^{a_{t+1}}$ by $< a_0, a_1, \ldots, a_{t-1} >$ and define $[a_0, \ldots, a_{t-1}] =_{df} < a_0 + 1, \ldots, a_{t-1} + 1 >$. Hence informally the finite sequence $\{a_0, \ldots, a_{t-1}\}$ is associated with the number (and finite sequence) $[a_0, \ldots, a_{t-1}]$, which is just $\bar{\alpha}(x)$, and $lh(\bar{\alpha}(x)) = t$. Indeed we see that our previous $< a_0, \ldots, a_{t-1} >$ is just $\tilde{\alpha}(x)$, and as before $lh(\tilde{\alpha}(x)) = t$.

In the same way we see that the two notions of concantenation agree if we restrict them to sequence numbers. The fundamental relation between sequence numbers is that of a sequence number a to the sequence number $a * 2^{s+1}$, which represents the sequence $\langle a_0, \ldots, a_{t-1}, s \rangle$, coming from $\langle a_0, \ldots, a_{t-1} \rangle$ by choosing one more s.

One function we have not as yet mentioned that does appear, is $\lambda t((u)_t \dot{-} 1)$ where $\mathrm{Seq}(u)$. If $u = p_0^{a_0} \ldots p_{t-1}^{a_{t-1}}$ then $\lambda t((u)_t \dot{-} 1)$ is $\langle a_0 \dot{-} 1, \ldots, a_{t-1} \dot{-} 1, 0, \ldots \rangle$. Informally it is $u * 0$, where 0 is the zero sequence. The important feature of this function is that for $z \leq lh(u)$, $\overline{\lambda t.((u)_t \dot{-} 1)}(z) = p_0^{a_0} \ldots p_{z-1}^{a_{z-1}}$.

4.2.9. Remarks. Although $\mathbf{BI!}$ looks at the one time weaker because of !, and stronger because it does not have $\forall u \forall w((\mathrm{Seq}(u) \wedge \mathrm{Seq}(w)) \to (R(u) \to R(u * w)))$ as a hypothesis, it is equivalent to the BI mentioned in Chapter One. For W.A. Howard and G. Kreisel in [7], show that $\mathbf{BI}$ with the second hypothesis $\forall u(\mathrm{Seq}(u) \to (R(u) \vee \neg R(u)))$ is equivalent to $\mathbf{BI}$ with the second hypothesis as above. Further S.C. Kleene in [10] shows that $\mathbf{BI!}$ is equivalent to $\mathbf{BI}$ with second hypothesis $\forall u(\mathrm{Seq}(u) \to (R(u) \vee \neg R(u)))$.

$\mathbf{BC-N!}$ is weaker than $\mathbf{BC-N}$, but J.R. Moschovakis has shown, in her Ph.D. Thesis 'Disjunction, existence and λ-definability in formalized intuitionistic analysis', that $\mathbf{BC-N!}$ is equivalent to $\mathbf{BC-F!}$. Notice also that the $\mathbf{BC-N!}$ of 4.2.6 has the second ! written out in full.

4.3. A TOPOLOGICAL MODEL OF INTUITIONISTIC SECOND ORDER ARITHMETIC.

4.3.1. Remark. Let $N = \{0, 1, 2, \ldots\}$, N^N be Baire Space, $C(N^N)$ the class of all continuous operators from N^N into N^N, N_I the class of all constant operators from N^N into N (which under the natural isomorphism can be identified with N).

Let $\mathcal{N}_I = (N_I, C(N^N), 0, {}^+, \times, f_4, \ldots)$ be the topological structure with domains N_I, $C(N^N)$. The functions $0, {}^+, +, \times, f_4, \ldots$ will be defined so that when evaluated at points in the Baire Space they are the usual primitive recursive functions, i.e. $0(\beta) = 0$, $x^y(\beta) = x(\beta)^{y(\beta)}$, etc, for each $\beta \in N^N$.

As before the assignments A are functions from the variables into the domains. However, because of our choice of domain for the number variables, we cannot extend assignments to functions from the set of terms into N_I. For example the term $\alpha\mathbf{x}$ would have to be sent to an element in N_I, but if we define $[\![\alpha\mathbf{x}]\!]_A(\beta) = [\![\alpha]\!]_A(\beta)([\![\mathbf{x}]\!]_A(\beta))$ as is natural, then because $[\![\alpha]\!]_A$ is not necessarily a constant operator, $[\![\alpha]\!]_A(\beta)([\![\mathbf{x}]\!]_A(\beta))$ is not necessarily a member of N_I. Hence such a definition would not be well-defined.

But since for our definition of valuation and for the subsequent propositions, we are only concerned with what is happening pointwise (e.g. $[\![\mathbf{a} = \mathbf{b}]\!]^T = \{\beta : [\![\mathbf{a}]\!]_A(\beta) = [\![\mathbf{b}]\!]_A(\beta)\}$), we get round this problem by restricting ourselves to what is happening pointwise. The idea is that if we are interested in $\beta \in [\![\mathbf{Ax}]\!]_A^T$, we take an $A(\mathbf{x})(\beta)$ that makes things work, and then define an assignment A' by $A'(x)(\gamma) = A(x)(\beta)$ for each γ.

Therefore we make this modification to the ideas of 4.1. Instead of the unique function from $\mathbf{T}$ into N_I, and the unique function from the functors into $C(N^N)$ (although there is no problem here in defining 'globally'), we define the unique function $[\![\ ]\!]_{A,\beta}$ from $\mathbf{T}$ into N and the unique function $[\![\ ]\!]_{A,\beta}$ from the functors into N^N by: For the functors: $[\![\alpha]\!]_{A,\beta} =_{df} A(\alpha)(\beta)$, $[\![\mathbf{f}_i]\!]_{A,\beta} =_{df} f_i(\beta)$, where $\mathbf{f}_i$ is a function of one number variable and no sequence variables; $[\![\lambda x.s]\!]_{A,\beta} =_{df} \lambda[\![x]\!]_{A^x_x,\beta} . [\![s]\!]_{A^x_x,\beta}$ where s is a term. For the terms: $[\![\mathbf{f}_i(t_1,\ldots, t_{k_i}, u_1,\ldots u_{l_i})]\!]_{A,\beta} =_{df}$ $f_i(\beta)([\![t_1]\!]_{A,\beta},\ldots, [\![u_{l_i}]\!]_{A,\beta})$, where $t_1,\ldots, t_{k_i}$ are terms, $u_1,\ldots, u_{l_i}$ are functors.

E.g. $[\![\bar{\alpha}s]\!]_{A,\beta} =_{df} \overline{[\![\alpha]\!]_{A,\beta}}([\![s]\!]_{A,\beta})$

$[\![u(a)]\!]_{A,\beta} =_{df} [\![u]\!]_{A,\beta}([\![a]\!]_{A,\beta})$, where u is a functor, a is a term.

For each assignment A we define $\mathcal{T}$-valuations by:

$[\![s=t]\!]^{\mathcal{T}} =_{df} \{\beta \in N^N : [\![s]\!]_{A,\beta} = [\![t]\!]_{A,\beta}\}$ for each term s, t (*);

$[\![A \wedge B]\!]^{\mathcal{T}} =_{df} [\![A]\!]^{\mathcal{T}} \cap [\![B]\!]^{\mathcal{T}}$, etc, as in 4.1.5.

By the above remarks we see that (*) is the same definition as in 4.1.5, so that the definition here also gives rise to open sets of N^N.

4.3.2. Remarks. In the presence of a wider class of functors, the usual induction on the complexity of formulas takes on a different form.

Let $A(s)$, $B(u)$ be predicates of terms s and functors u.

If $A(\mathbf{0})$ and $A(x)$, for each variable x, $B(f_i)$, for each f_i a function of one number and no function variables, and $B(\alpha)$, for each variable α, $A(s)$ implies $B(\lambda x.s)$, for each variable x and term s, $A(s)$ and $B(u)$ imply $A(u(s))$, for each term s and functor u, $A(t_1),\ldots, A(t_{k_i})$, $B(u_1),\ldots, B(u_{l_i})$ imply $A(f_i(t_1,\ldots, t_{k_i}, u_1,\ldots, u_{l_i}))$ for each term $t_1,\ldots, t_{k_i}$, functor $u_1,\ldots, u_{l_i}$, then $A(s)$ and $B(u)$ for each term s and functor u. A double induction.

On the other hand propositions $u = v$ introduce no new features at all, since $u = v$ is an abbreviation of $\forall x(u(x) = v(x))$ and $u(x) = v(x)$ and $\forall x$ are covered as before. I.e. given a predicate C on formulas, if $C(s = t)$ for each term s, t,

$C(A)$ implies $C(\neg A)$, $C(\forall x A)$, $C(\forall\alpha A),\ldots$ for all variables x, α,

$C(A)$ and $C(B)$ imply $C(A \vee B)$, $C(A \wedge B),\ldots$ for each A, B, then $C(A)$ for each formula A.

4.3.3. Remark. To show that the intuitionistic propositional calculus axiom schemas are valid in N_I is straightforward, e.g. for modus ponens for each A, $[\![A]\!]^T_A = N^N$, $[\![A \to B]\!]^T_A = N^N$

imply $[\![A]\!]^T_A = N^N$ and $((\text{int-}[\![A]\!]^T_A) \cup [\![B]\!]^T_A) = N^N$

imply $-[\![A]\!]^T_A = \phi$ and $((\text{int-}[\![A]\!]^T_A) \cup [\![B]\!]^T_A) = N^N$

imply $\text{int-}[\![A]\!]^T_A = \phi$ and $((\text{int-}[\![A]\!]^T_A) \cup [\![A]\!]^T_A) = N^N$

imply $[\![B]\!]^T_A = N^N$;

i.e. if A and $A \to B$ then B is valid in N_I.

4.3.4. Remarks. To show that the intuitionistic predicate calculus axiom schemas are valid in N_T we need to prove the following lemmas.

Because λ binds variables we have to be careful that when we substitute $\mathbf{b}$ for $\mathbf{x}$ in a term or functor which contains λ, we do not go from a valid to an invalid result, e.g. $\{\lambda\mathbf{y}.\mathbf{x}\}(\mathbf{a}) = \mathbf{x}$ is valid, but $S^{\mathbf{y}}_{\mathbf{x}}(\{\lambda\mathbf{y}.\mathbf{x}\}(\mathbf{a}) = \mathbf{x})$ is $\{\lambda\mathbf{y}.\mathbf{y}\}(\mathbf{a}) = \mathbf{y}$ or $\mathbf{a} = \mathbf{y}$, which in general is invalid. Hence we introduce the notion of substitutability analogous to that for the case of $\exists,\forall$ for formulas. Term $\mathbf{b}$ is substitutable for $\mathbf{x}$ in term $\mathbf{s}$ or functor $\mathbf{u}$ $=_{df}$ $\mathbf{x}$ does not occur free within the scope of an occurrence of $\lambda\mathbf{y}$ for any $\mathbf{y}$ which occurs in $\mathbf{b}$. Then $S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{s})$ is the term obtained from $\mathbf{s}$ by replacing each free occurrence of $\mathbf{x}$ by an occurrence of $\mathbf{b}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{u}$.

4.3.5. Lemma. For each assignment A, for each variable $\mathbf{x}$, terms $\mathbf{a}$, $\mathbf{b}$, functor $\mathbf{u}$,

$$[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{a})]\!]_{A,\beta} = [\![\mathbf{a}]\!]_{A^{\lambda\gamma.[\![\mathbf{b}]\!]_{A,\beta}}_{\mathbf{x}},\beta} \quad \text{, if } \mathbf{b} \text{ is substitutable for } \mathbf{x} \text{ in } \mathbf{a},$$

$$[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{u})]\!]_{A,\beta} = [\![\mathbf{u}]\!]_{A^{\lambda\gamma.[\![\mathbf{b}]\!]_{A,\beta}}_{\mathbf{x}},\beta} \quad \text{, if } \mathbf{b} \text{ is substitutable for } \mathbf{x} \text{ in } \mathbf{u},$$

For each assignment A, for each variable $\mathbf{x}$, term $\mathbf{b}$, formula $\mathbf{A}$, $\beta \in [\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{A})]\!]^{T}_{A}$ iff $\beta \in [\![\mathbf{A}]\!]^{T}_{A^{\lambda\gamma.[\![\mathbf{b}]\!]_{A,\beta}}_{\mathbf{x}}}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{A}$.

Proof. By an induction on the complexity of terms and formulas. In the 'induction basis' for the statement about terms and functors:

for each A, for the variable $\mathbf{x}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{x}$ then $S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{x})$ is $\mathbf{b}$ and $[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{x})]\!]_{A,\beta} = [\![\mathbf{b}]\!]_{A,\beta}$

$$= [\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}](\beta)$$

$$= A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}(\mathbf{x})(\beta)$$

$$= [\![\mathbf{x}]\!]_{A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}, \beta};$$

for the variable $\mathbf{y}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{y}$ then $S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{y})$ is $\mathbf{y}$ and $[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{y})]\!]_{A,\beta} = [\![\mathbf{y}]\!]_{A,\beta}$

$$= A(\mathbf{y})(\beta)$$

$$= A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}(\mathbf{y})(\beta)$$

$$= [\![\mathbf{y}]\!]_{A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}, \beta}.$$

Similar for $\mathbf{0}$.

For the functor $^{+}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $^{+}$ then $S^{\mathbf{b}}_{\mathbf{x}}(^{+})$ is $^{+}$ and $[\![S^{\mathbf{b}}_{\mathbf{x}}(^{+})]\!]_{A,\beta} = [\![^{+}]\!]_{A,\beta}$

$$= {}^{+}(\beta)$$

$$= [\![^{+}]\!]_{A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}, \beta}.$$

For the functor α, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in α then $S^{\mathbf{b}}_{\mathbf{x}}(\alpha)$ is α and $[\![S^{\mathbf{b}}_{\mathbf{x}}(\alpha)]\!]_{A,\beta} = [\![\alpha]\!]_{A,\beta}$

$$= A(\alpha)(\beta)$$

$$= A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}(\alpha)(\beta)$$

$$= [\![\alpha]\!]_{A_{\mathbf{x}}^{\lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}, \beta}.$$

Similarly for the other $\mathbf{f}_i$ functions of one number variable and no sequence variable. In the 'induction step' for the statement about terms and functors:

For each A, for each term $\mathbf{s}$, for the variable $\mathbf{x}$, for the functor $\lambda\mathbf{x}\mathbf{s}$, if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{s}$ implies

$[\![S_x^b(s)]\!]_{A,\beta} = [\![s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}},\beta}$ then b is substitutable for x in $\lambda x s$ implies $S_x^b(\lambda x s)$ is $\lambda x s$ and

$$[\![S_x^b(\lambda x s)]\!]_{A,\beta} = [\![\lambda x s]\!]_{A,\beta}$$

$$= \lambda [\![x]\!]_{A_x^x,\beta} \cdot [\![s]\!]_{A_x^x,\beta}$$

$$= \lambda [\![x]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_x^x,\beta} \cdot [\![s]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_x^x,\beta}$$

[since $[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_x^x$ is A_x^x]

$$= [\![\lambda x s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}},\beta} .$$

For the variable y, for the functor $\lambda y s$, if b is substitutable for x in s implies $[\![S_x^b(s)]\!]_{A,\beta} = [\![s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}},\beta}$ then b is substitutable for x in $\lambda y s$ implies b is substitutable for x in s and $S_x^b(\lambda y s)$ is $\lambda y\ S_x^b(s)$ and $[\![S_x^b(\lambda y s)]\!]_{A,\beta} = [\![\lambda y\ S_x^b(s)]\!]_{A,\beta}$

$$= \lambda [\![y]\!]_{A_y^y,\beta}\ [\![S_x^b(s)]\!]_{A_y^y,\beta}$$

$$= \lambda [\![y]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_y^y,\beta} \cdot [\![S_x^b(s)]\!]_{A_y^y,\beta}$$

[since $[\![y]\!]_{A_y^y,\beta} = [\![y]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_y^y,\beta}$]

$$= \lambda [\![y]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_y^y,\beta} \cdot [\![s]\!]_{[A_y^y]_x^{\lambda\gamma . [\![b]\!]_{A_y^y,\beta}},\beta}$$

[since b is substitutable for x in s and hypothesis applied to A_y^y]

$$= \lambda [\![y]\!]_{[A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}}]_y^y,\beta} \cdot [\![s]\!]_{[A_y^y]_x^{\lambda\gamma . [\![b]\!]_{A,\beta}},\beta}$$

[since b is substitutable for x in $\lambda y s$ implies y does not occur in b so that $[\![b]\!]_{A_y^y,\beta} = [\![b]\!]_{A,\beta}$]

$$= \lambda \| y \|_{[A_x^{\lambda\gamma . \| b \|_{A,\beta}}]_y^y , \beta} \cdot \| s \|_{[A_x^{\lambda\gamma . \| b \|_{A,\beta}}]_y^y , \beta}$$

[since y is not x]

$$= [\![\lambda y . s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta} .$$

For each A, for each term s, each functor u, for the term $u(s)$, if b is substitutable for x in s implies $[\![S_x^b(u)]\!]_{A,\beta} = [\![u]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta}$ if b is substitutable for x in $u(s)$ implies b is substitutable for x in s and b is substitutable for x in u and

$S_x^b(u(s))$ is $S_x^b(u) S_x^b(s)$ and

$$[\![S_x^b(u(s))]\!]_{A,\beta} = [\![S_x^b(u) S_x^b(s)]\!]_{A,\beta}$$

$$= [\![S_x^b(u)]\!]_{A,\beta} \, ([\![S_x^b(s)]\!]_{A,\beta})$$

$$= [\![u]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta} \, ([\![s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta})$$

[since b is substitutable for x in s and u and by hypothesis].

$$= [\![u(s)]\!]_{A_x^{\lambda\gamma [\![b]\!]_{A,\beta}} , \beta} .$$

For each A, for all terms s, t, for the term $s + t$, if b is substitutable for x in s implies $[\![S_x^b(s)]\!]_A = [\![s]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta'}$ and b is substitutable for x in t implies $[\![S_x^b(t)]\!]_A = [\![t]\!]_{A_x^{\lambda\gamma . [\![b]\!]_{A,\beta}} , \beta}$, then b is substitutable for x in $s+t$ implies b is substitutable for x in s and b is substitutable for x in t and

$S_x^b(s + t)$ is $S_x^b(s) + S_x^b(t)$ and

$$[\![S_x^b(s + t)]\!]_{A,\beta} = [\![S_x^b(s) + S_x^b(t)]\!]_{A,\beta}$$

$$= [\![S_x^b(s)]\!]_{A,\beta} + [\![S_x^b(t)]\!]_{A,\beta}$$

$$= [\![s]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta} + [\![t]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta}$$

[since b is substitutable for x in s and t and by hypothesis]

$$= [\![s + t]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta} .$$

For each A, for each term s, for each functor u, for the term $\bar{u}(s)$, if b is substitutable for x in s implies $[\![S_x^b(s)]\!]_{A,\beta} = [\![s]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta}$, and b is substitutable for x in u implies $[\![S_x^b(u)]\!]_{A,\beta} = [\![u]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta}$ then b is substitutable for x in $\bar{u}(s)$ implies b is substitutable for x in u and b is substitutable for x in s and $S_x^b(\bar{u}(s))$ is $S_x^b(\Pi_{i<s}\ p_i^{u(i)+1})$ is $\Pi_{i<S_x^b(s)}\ p_i^{S_x^b(u(i)+1)}$ is

$\Pi_{i<S_x^b(s)}\ p_i^{S_x^b(u(i))+1}$ is $\overline{S_x^b(u)}(S_x^b(s))$

and $[\![S_x^b(\bar{u}(s))]\!]_{A,\beta} = [\![\overline{S_x^b(u)}(S_x^b(s))]\!]_{A,\beta}$

$$= \overline{[\![S_x^b(u)]\!]_{A,\beta}}\ ([\![S_x^b(s)]\!]_{A,\beta})$$

$$= \overline{[\![u]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta}}\ ([\![s]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta})$$

[since b is substitutable for x in s and u and by hypothesis]

$$= [\![\bar{u}(s)]\!]_{A_x^{\lambda\gamma.[\![b]\!]_{A,\beta}},\beta} .$$

Similarly for the other 'primitive recursive' functions. In the 'induction basis' for the statement about formulas, for each A, for $a = c$, if b is substitutable for x in $a = c$ then b is substitutable for x in a and b is substitutable for x in c and so $S_x^b(a = c)$ is $S_x^b(a) = S_x^b(c)$. Then

$\beta \in \| S_x^b(a = c) \|_A^T$ iff $\beta \in \| S_x^b(a) = S_x^b(c) \|_A^T$

iff $\beta \in \{\beta \in N^N : \| S_x^b(a) \|_{A,\beta} = \| S_x^b(c) \|_{A,\beta}\}$

iff $\| S_x^b(a) \|_{A,\beta} = \| S_x^b(c) \|_{A,\beta}$

iff $\| a \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}},\beta} = \| c \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}},\beta}$

[since b is substitutable for x in a and c and by first part of lemma]

iff $\beta \in \{\beta \in N^N : \| a \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}},\beta} = \| c \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}},\beta}$

iff $\beta \in \| a = c \|^T_{A_x^{\lambda\gamma.\| b \|_{A,\beta}}}$.

In the 'induction step' for the statement about formulas, for each A, for the variable x, for $\forall x$, if b is substitutable for x in A implies $\beta \in \| S_x^b(A) \|_A^T$ iff $\beta \in \| A \|_{A_x^{\lambda\gamma\ \| b \|_{A,\beta}}}$, then b is substitutable for x in $\forall x A$ implies

$S_x^b(\forall x A)$ is $\forall x A$ and

$\beta \in \| S_x^b(\forall x A) \|_A^T$ iff $\beta \in \| \forall x A \|_A^T$

iff $\beta \in \text{int} \cap \{\| A \|_{A_x^x} : x \in N_I\}$

iff $\beta \in \text{int} \cap \{\| A \|^T_{[A_x^{\lambda\gamma.\| b \|_{A,\beta}}]_x^x} : x \in N_I\}$

[since $[A_x^{\lambda\gamma.\| b \|_{A,\beta}}]_x^x$ is A_x^x]

iff $\beta \in \| \forall x A \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}}}$.

For the variable y, for $\forall y$, if b is substitutable for x in A implies $\beta \in \| S_x^b(A) \|_A^T$ iff $\beta \in \| A \|_{A_x^{\lambda\gamma.\| b \|_{A,\beta}}}$, then b is substitutable for x in $\forall y A$ implies b is substitutable for x in A and $S_x^b(\forall y A)$ is $\forall y S_x^b(A)$ and

$\beta \in [\![\mathbf{S}_{\mathbf{x}}^{\mathbf{b}}(\forall \mathbf{y} \mathbf{A})]\!]_{A}^{T}$ iff $\beta \in [\![\forall \mathbf{y} \mathbf{S}_{\mathbf{x}}^{\mathbf{b}}(\mathbf{A})]\!]_{A}^{T}$

iff $\beta \in \text{int} \cap \{[\![\mathbf{S}_{\mathbf{x}}^{\mathbf{b}}(\mathbf{A})]\!]_{A_{\mathbf{y}}^{y}}^{T} : y \in N_I\}$

iff $\beta \in \text{int} \cap \{[\![\mathbf{A}]\!]_{[A_{\mathbf{y}}^{y}]} \lambda\gamma . [\![\mathbf{b}]\!]_{A_{\mathbf{y}}^{y}, \beta} : y \in N_I\}$

[since **b** is substitutable for **x** in **A** and using hypothesis applied to $A_{\mathbf{y}}^{y}$]

iff $\beta \in \text{int} \cap \{[\![\mathbf{A}]\!]^{T}_{[A_{\mathbf{y}}^{y}]} \lambda\gamma . [\![\mathbf{b}]\!]_{A, \beta} : y \in N_I\}$

[since **b** is substitutable for **x** in $\forall \mathbf{y} \mathbf{A}$ implies **y** does not occur in **b** so that $[\![\mathbf{b}]\!]_{A_{\mathbf{y}}^{y}, \beta} = [\![\mathbf{b}]\!]_{A, \beta}$]

iff $\beta \in \text{int} \cap \{[\![\mathbf{A}]\!]^{T}_{[A_{\mathbf{x}} \lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}]_{\mathbf{y}}^{y}} : y \in N_I\}$

[since **y** is not **x**]

iff $\beta \in [\![\forall \mathbf{y} \mathbf{A}]\!]^{T}_{A_{\mathbf{x}} \lambda\gamma . [\![\mathbf{b}]\!]_{A,\beta}}$.

Similar and sometimes easier arguments hold for the other quantifiers and logical connections.

<u>4.3.6. Lemma</u>. For each assignment A, for each variable α, term **a**, functor **u**, **v**:

$[\![\mathbf{S}_{\alpha}^{\mathbf{u}}(\mathbf{a})]\!]_{A,\beta} = [\![\mathbf{a}]\!]_{A_{\alpha} \lambda\gamma . [\![\mathbf{u}]\!]_{A,\gamma}, \beta}$ where $\mathbf{S}_{\alpha}^{\mathbf{u}}(\mathbf{a})$ is the term obtained from **a** by replacing each (free) occurrence of α by an occurrence of **u**:

$[\![\mathbf{S}_{\alpha}^{\mathbf{u}}(\mathbf{v})]\!]_{A,\beta} = [\![\mathbf{v}]\!]_{A_{\alpha} \lambda\gamma . [\![\mathbf{u}]\!]_{A,\gamma}, \beta}$ where $\mathbf{S}_{\alpha}^{\mathbf{u}}(\mathbf{v})$ is a functor obtained from **v** by replacing each (free) occurrence of α by an occurrence of **u**.

For each assignment A, for each variable α, functor $\mathbf{u}$, formula $\mathbf{A}$, $[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{A})]\!]^{T}_{A} = [\![\mathbf{A}]\!]^{T}_{A_{\alpha}^{\lambda\gamma.[\![\mathbf{u}]\!]_{A,\gamma}}}$ if $\mathbf{u}$ is substitutable for α in $\mathbf{A}$.

<u>4.3.7. Remarks.</u> The proof of 4.3.6 is by an induction on the complexity of terms and formulas. It is very similar to 4.3.5. Note, however, that on the RHS the assignment is $A_{\alpha}^{\lambda\gamma.[\![\mathbf{u}]\!]_{A,\gamma}}$, not $A_{\alpha}^{\lambda\gamma.[\![\mathbf{u}]\!]_{A,\beta}}$. This is because the domain to which the sequence variables are sent is $C(N^N)$, not the constant functions on N^N. If our domain to which the number variables are sent was the continuous functions from N^N into N (rather than the constant functions) then we could strengthen 4.3.5 to $[\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{A})]\!]^{T}_{A} = [\![\mathbf{A}]\!]^{T}_{A_{\mathbf{x}}^{\lambda\gamma.[\![\mathbf{b}]\!]_{A,\gamma}}}$ in the manner of 4.3.6. It is to ensure that we are assigning constant functions from N^N into N to our number variables that we have to weaken 4.3.5.

<u>4.3.8. Proposition.</u> If $\mathbf{A}$ is an intuitionistic predicate calculus axiom schema then $\mathbf{A}$ is valid in N_I.

<u>Proof.</u> Using 4.3.5, 4.3.6, e.g. **2N**,

for each A, $\beta \in [\![\forall \mathbf{x}\, \mathbf{Ax}]\!]^{T}_{A}$ implies $\beta \in \operatorname{int} \cap \{[\![\mathbf{Ax}]\!]^{T}_{A^x_{\mathbf{x}}} : x \in N_I\}$

implies $\beta \in \cap \{[\![\mathbf{Ax}]\!]^{T}_{A^x_{\mathbf{x}}} : x \in N_I\}$

implies $\beta \in [\![\mathbf{Ax}]\!]^{T}_{A^x_{\mathbf{x}}}$ for each $x \in N_I$

implies $\beta \in [\![\mathbf{Ax}]\!]^{T}_{A_{\mathbf{x}}^{\lambda\gamma.[\![\mathbf{b}]\!]_{A,\beta}}}$ and $\beta \in [\![\mathbf{Ax}]\!]^{T}_{A}$

implies $\beta \in [\![S^{\mathbf{b}}_{\mathbf{x}}(\mathbf{Ax})]\!]^{T}_{A}$ if $\mathbf{b}$ is substitutable for $\mathbf{x}$ in $\mathbf{A}$, and $\beta \in [\![\mathbf{Ax}]\!]^{T}_{A}$

implies $\beta \in [\![S_x^b(Ax)]\!]_A^T$

[since if b is not substitutable for x in Ax then $S_x^b(Ax)$ is Ax],

i.e. $[\![\forall x Ax \rightarrow S_x^b(Ax)]\!]_A^T = \text{int } (-[\![\forall x Ax]\!]_A^T) \cup [\![S_x^b(Ax)]\!]_A^T = N^N$.

4.3.9. Remark. Note that if a formula A contains no sequence variables then, because of the assignment of number variables to constant functions from N^N into N, $[\![A]\!]_A^T$ is either N^N or ϕ. Thus this assignment of number variables ensures that the axiom schemas for intuitionistic number theory are valid in N_I.

The axiom schema **PIn** does not seem to hold in any topological structure larger than the smallest one, which is why we have N_I instead of the continuous functions from N^N into N as a domain to which the number variables are assigned.

The other axiom schema are trivially valid by our choice of $^+$, $+$, $\times$, $f_4, \ldots$. **PIn** is shown to be valid using 4.3.5, and the isomorphism between N_I and N, and informal induction.

4.3.10. Remark. $\lambda 1$ and $=F$ are trivially valid in N_I because of the inductive definition of $[\![\,]\!]_{A,\beta}$. For example,

$$[\![(\lambda x s)(t)]\!]_{A,\beta} = (\lambda [\![x]\!]_{A_x^x,\beta} . [\![s]\!]_{A_x^x,\beta})([\![t]\!]_{A,\beta})$$

$$= (\lambda x(\beta) . [\![s]\!]_{A_x^x,\beta})([\![t]\!]_{A,\beta})$$

$$= [\![s]\!]_{A_x^{\lambda\gamma . [\![t]\!]_{A,\beta}},\beta}$$

[since $[\![t]\!]_{A,\beta} = (\lambda\gamma . [\![t]\!]_{A,\beta})(\beta)$]

$= \| S^t_x(s) \|_{A,\beta}$ if t is substitutable for x in s,

so that $[\![(\lambda x s)(t) = S^t_x(s)]\!]^T_A = N^N$ if t is substitutable for x in s, for each A.

4.3.11. Lemma. Suppose $A(\alpha)$ is any formula, Φ, $\Psi \in C(N^N)$. Suppose $\beta \in \{\gamma : \Phi(\gamma) = \Psi(\gamma)\}$, then $\beta \in [\![A(\alpha)]\!]^T_{A^\Phi_\alpha}$ iff $\beta \in [\![A(\alpha)]\!]^T_{A^\Psi_\alpha}$

Proof. By induction on the complexity of $A(\alpha)$, for 'the induction basis' when $A(\alpha)$ is $a(\alpha) = b(\alpha)$ for terms a, b, we need to show $[\![a(\alpha)]\!]_{A^\Phi_\alpha,\beta} = [\![a(\alpha)]\!]_{A^\Psi_\alpha,\beta}$,
$[\![b(\alpha)]\!]_{A^\Phi_\alpha,\beta} = [\![b(\alpha)]\!]_{A^\Psi_\alpha,\beta}$ if $\beta \in \{\gamma : \Phi(\gamma) = \Psi(\gamma)\}$.

This is just a lengthy proof of the type in 4.3.5. In view of this we shall not give the proof of this lemma.

4.3.12. Proposition. The axiom schema **AC** is valid in N_I.

Proof. Let A be any assignment. Let $\beta \in N^N$.
If $\beta \in [\![\forall x \exists \alpha A(x,\alpha)]\!]^T_A$ then there is a neighbourhood V_0 of β such that $V_0 \subseteq [\![\forall x \exists \alpha A(x,\alpha)]\!]^T_A$,
i.e. $V_0 \subseteq \operatorname{int} \cap \{[\![\exists \alpha A(x,\alpha)]\!]^T_{A^x_x} : x \in N_I\} \subseteq \cap\{[\![\exists \alpha A(x,\alpha)]\!]^T_{A^x_x} : x \in N_I\}$.
Hence for each $x \in N_I$, $V_0 \subseteq \cup\{[\![A(x,\alpha)]\!]^T_{A^{x,\Phi}_{x,\alpha}} : \Phi \in C(N^N)\}$.
Since for each $x \in N_I$, $\cup\{[\![A(x,\alpha)]\!]_{A^{x,\Phi}_{x,\alpha}} : \Phi \in C(N^N)\}$ is open, it is the countable union of disjoint clopen neighbourhoods, by 1.1.7 for Baire Space.

Therefore for each $x \in N_I$, the clopen set V_0 can be partitioned into countably many disjoint clopen neighbourhoods V^x_1, $V^x_2, \ldots$, namely the intersection of V_0 and members of the countable union of disjoint clopen neighbourhoods of

$\cup\{[\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \Phi : \Phi \in C(N^N)\}$, such that for each $\gamma \in V_0$, $\gamma \in V_j^x$ for exactly one j. Moreover, since for each $x \in N_I$, $V_0 \subseteq \cup\{[\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \Phi : \Phi \in C(N^N)\}$, for each $x \in N_I$, for each γ, $\gamma \in V_j^x$ implies there is some $\Phi \in C(N^N)$ such that $\gamma \in [\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \Phi$. Since there is no Φ in $\gamma \in V_j^x$ we have, for each $x \in N_I$, for each γ, there is some $\Phi \in C(N^N)$ such that $\gamma \in V_j^x$ implies $\gamma \in [\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \Phi$. By the axiom of choice we have, for each $x \in N_I$, there is some $\Phi_j^x \in C(N^N)$ such that for each γ, $\gamma \in V_j^x$ implies $\gamma \in [\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \Phi_j^x$, (1)

For each $\gamma \in N^N$, each $n \in N$, define Ψ as follows,

$$[\Psi(\gamma)](n) =_{df} [\Phi_j^x(\gamma)](y) \quad \text{if } \gamma \in V_j^x \text{ and } n = 2^x . 3^y,$$

$$=_{df} 0 \quad \text{otherwise.}$$

Because each $\Phi_j^x \in C(N^N)$ and V_0 is clopen, this defines a continuous functional, i.e. $\Psi \in C(N^N)$. Moreover on V_0, $\lambda y.[\Psi(\gamma)](2^x . 3^y) = \Phi_j^x(\gamma)$, where j is given by $\gamma \in V_j^x$.

But since $x, y \in N$ are identified with the constant functions from $N^N \to N$ defined by $x(\beta) = x$, $y(\beta) = y$, for each β, we have on V_0,

$$\lambda\gamma . \lambda y(\gamma) . [\Psi(\gamma)](2^{x(\gamma)} . 3^{y(\gamma)}) = \Phi_j^x$$

or $\lambda\gamma . \lambda y(\gamma) \; [\Psi(\gamma)](2^x , 3^y)(\gamma) \quad = \Phi_j^x$ [by definition of our 'primitive recursive' functions]. Hence in view of (1), by 4.3.6 lemma we have, for each $x \in N_I$, for each γ, $\gamma \in V$ implies $\gamma \in V_j^x$, where j is given by $\gamma \in V_j^x$

implies $\gamma \in [\![\mathbf{A}(x,\alpha)]\!]^{T}_{A_{x,\alpha}} x, \lambda\gamma . \lambda y(\gamma) [\Psi(\gamma)](2^x . 3^y)(\gamma)$ - (a)

implies $\gamma \in [\![\mathbf{A}(x,\alpha)]\!]^{T}_{[A_{x,\alpha}} x, \lambda\gamma\Psi(\gamma)]_{\alpha} \lambda\gamma . \lambda y(\gamma) [\Psi(\gamma)](2^x . 3^y)(\gamma)$

implies $\gamma \in \| A(x,\alpha) \|_{[A^{x,\lambda\gamma\Psi(\gamma)}_{x,\alpha}]^{\lambda\gamma.\| \lambda y\,\alpha(2^x 3^y) \|_{A^{x,\lambda\gamma\Psi(\gamma)}_{x,\alpha},\gamma}}_{\alpha}}$

[since $y(\gamma) = [\![y]\!]_{A^{x,y,\lambda\gamma\Psi(\gamma)}_{x,y,\alpha},\gamma}$, $\Psi(\gamma) = [\![\alpha]\!]_{A^{x,y,\lambda\gamma\Psi(\gamma)}_{x,y,\alpha},\gamma}$, $(2^x.3^y)(\gamma) = [\![2^x 3^y]\!]_{A^{x,y,\lambda\gamma\Psi(\gamma)}_{x,y,},\gamma}$, and by definition of $[\![\lambda x s]\!]_{A,\gamma}$]

implies $\gamma \in [\![S^{\lambda y\,\alpha(2^x 3^y)}_{\alpha}(A(x,\alpha))]\!]^{T}_{A^{x,\lambda\gamma.\Psi(\gamma)}_{x,\alpha}}$

[by 4.3.6, since x is free for α in $A(x,\alpha)$ implies $\lambda y\,\alpha(2^x 3^y)$ is substitutable for α in $A(x,\alpha)$]

implies $\gamma \in [\![A(x,\lambda y.\alpha(2^x 3^y))]\!]^{T}_{A^{x,\lambda\gamma\Psi(\gamma)}_{x,\alpha}}$ — (b)

Therefore since int $V_0 \subseteq V_0$,

$V_0 \subseteq [\![\forall x A(x,\lambda y.\alpha(2^x 3^y))]\!]^{T}_{A^{\lambda\gamma\Psi(\gamma)}_{\alpha}}$.

Therefore $V_0 \subseteq [\![\exists\alpha\,\forall x A(x,\lambda y\,\alpha(2^x 3^y))]\!]^{T}_{A}$.

Therefore $[\![\forall x \exists\alpha A(x,\alpha) \rightarrow \exists\alpha\forall x A(x,\lambda y\,\alpha(2^x 3^y))]\!]^{T}_{A} = N^N$ if x is free for α in $A(x,\alpha)$.

4.3.13. Remark. To show **BI!** is valid in N_I we need the following two lemmas.

4.3.14. Lemma. Let $A(x)$ be any formula, let A be any assignment. Then $[\![\exists! x Ax]\!]^{T}_{A}$ is a disjoint union of sets $[\![Ax]\!]^{T}_{A^x_x}$ which are clopen in $[\![\exists! x Ax]\!]^{T}_{A}$.

Proof. For each $\gamma \in N^N$, $\gamma \in [\![\exists! Ax]\!]^{T}_{A}$

implies $\gamma \in [\![\exists x(Ax \wedge \forall y(Ay \rightarrow x = y))]\!]^{T}_{A}$ where y does not occur in Ax

implies $\gamma \in [\![Ax \wedge \forall y(Ay \rightarrow x = y)]\!]^{T}_{A^x_x}$ for some $x \in N_I$

implies $\gamma \in [\![Ax]\!]_{A^x_x}$ and $\gamma \in [\![\forall y(Ay \rightarrow x = y)]\!]^{T}_{A^x_x}$ for some $x \in N_I$

implies $\gamma \in [\![Ax]\!]^{T}_{A_x} x$ and $\gamma \in [\![Ay \to x = y]\!]^{T}_{A_{x,y}} x,y$ for each

$y \in N_I$, for some $x \in N_I$

[since int $X \subseteq X$ for each $X \subseteq N^N$]

implies $\gamma \in [\![Ax]\!]^{T}_{A_x} x$ and [not $\gamma \in [\![Ay]\!]^{T}_{A_{x,y}} x,y$ or $\gamma \in [\![x=y]\!]^{T}_{A_{x,y}} x,y$]

for each $y \in N_I$, for some $x \in N_I$

implies $\gamma \in [\![Ax]\!]^{T}_{A_x} x$ and for each $y \in N_I$, if $x \neq y$ then

$\gamma \notin [\![Ay]\!]^{T}_{A_{x,y}} x,y$ for some $x \in N_I$

implies $\gamma \in [\![Ax]\!]^{T}_{A_x} x$ and for each $y \in N_I$, if $x \neq y$ then

$\gamma \notin [\![Ay]\!]^{T}_{A_y} y$ for some $x \in N_I$

[since x does not occur free in Ay]

implies $\gamma \in [\![Ax]\!]^{T}_{A_x} x$ and for each $y \in N_I$, if $x \neq y$ then

$\gamma \notin [\![Ax]\!]^{T}_{A_x} y$ for some $x \in N_I$

[since y does not occur in Ax] .

Therefore $[\![\exists ! Ax]\!]^{T}_{A}$ is a disjoint union of open sets $[\![Ax]\!]^{T}_{A_x} x$.

For each $[\![Ax]\!]^{T}_{A_x} x$, the complement of $[\![Ax]\!]^{T}_{A_x} x$ in $[\![\exists ! x Ax]\!]_{A}$ is $\cup\{[\![Ax]\!]^{T}_{A_x} y : y \neq x\}$, which is the union of open sets and so is open, and open in $[\![\exists ! x Ax]\!]^{T}_{A}$.

<u>4.3.15. Lemma</u>. Suppose $\beta \in [\![\forall \alpha \exists ! x R(\bar{\alpha}x) \wedge \forall w(Seq(w) \to (Rw \to Aw)) \wedge \forall w(Seq(w) \to (\forall s A(w * 2^{s+1}) \to Aw))]\!]^{T}_{A}$, then for each $w \in N_I$, $\beta \notin [\![Seq(w) \to Aw]\!]^{T}_{A_w} w$ and $\beta \notin [\![(\exists z \leqslant lh(w))(Seq(w) \to R(\overline{\lambda t(w)_t \dot{-} 1}(z)))]\!]^{T}_{A_w} w$

implies for some $u \in N_I$,

$\beta \notin [\![(\exists z \leqslant lh(w*u))((Seq(w) \wedge Seq(u))$

$$\rightarrow R(\overline{\lambda t(w)_t \dot{-} 1(z)})) \rrbracket^{T}_{A^w_w}$$

and $\beta \in \cap\{\llbracket (Seq(w) \wedge Seq(u) \rightarrow A(w*u*2^{s+1}) \rrbracket^{T}_{A^{w,u,s}_{w,u,s}} : s \in N_I\}$

and $\beta \notin int \cap\{\llbracket (Seq(w) \wedge Seq(u)) \rightarrow A(w*u*2^{s+1}) \rrbracket^{T}_{A^{x,u,s}_{x\ u\ s}} : s \in N_I\}$

Proof. The proof is by contradiction. We assume the opposite of the lemma, as follows:

Suppose $\beta \in \llbracket \forall\alpha\exists! xR(\bar{\alpha}x) \wedge \forall w(Seq(w) \rightarrow (Rw \rightarrow Aw)) \wedge$

$$\forall w(Seq(w) \rightarrow (\forall s A(w*2^{s+1}) \rightarrow Aw)) \rrbracket^{T}_{A}$$

and for some $w \in N_I$, $\beta \notin \llbracket Seq(w) \rightarrow Aw \rrbracket^{T}_{A^w_w}$ - ①

and $\beta \notin \llbracket (\exists z \leqslant lh(w))(Seq(w) \rightarrow R(\overline{\lambda t(w)_t \dot{-} 1(s)})) \rrbracket^{T}_{A^w_w}$ - ②

and ③ for each $u \in N_I$,

[$\beta \notin \llbracket (\exists z \leqslant lh(w*u))((Seq(w) \wedge Seq(u))$

$\rightarrow R(\overline{\lambda t(w*u)_t \dot{-} 1(z)})) \rrbracket^{T}_{A^{w,u}_{w,u}}$

and $\beta \in \cap\{\llbracket (Seq(w) \wedge Seq(w)) \rightarrow A(w*u*2^{s+1}) \rrbracket^{T}_{A^{w,u,s}_{w,u,s}} : s \in N_I\}$]

implies $\beta \in int \cap\{\llbracket (Seq(w) \wedge Seq(u))$

$\rightarrow A(w*u*2^{s+1}) \rrbracket^{T}_{A^{w,u,s}_{w,u,s}} : s \in N_I\}$.

Suppose for some $w \in N_I$ satisfying the assumption,

$\beta \in \cap\{\llbracket Seq(w) \rightarrow A(w*2^{s+1}) \rrbracket^{T}_{A^{w,s}_{w,s}} : s \in N_I\}$ - ④

Since $1 = \lambda\gamma.\llbracket 1 \rrbracket_{A^w_w,\beta} = \lambda\gamma.\llbracket 1 \rrbracket_{A^{w,s}_{w,s},\beta}$ is in N_I, and since 1 is a closed term and so substitutable for u in each formula A, using 4.3.5 ③ implies

[$\beta \notin \llbracket S^1_u((\exists z \leqslant lh(w*u))((Seq(w) \wedge Seq(u))$

$\rightarrow R(\overline{\lambda t(w*u)_t \dot{-} 1(z)}))) \rrbracket^{T}_{A^{w,u}_{w,u}}$

$$\text{and } \beta \in \cap\{[\![S^1_u((Seq(w) \wedge Seq(u)) \rightarrow A(w*u*2^{s+1}))]\!]^T_{A^{w,u,s}_{w,u,s}} : s \in N_I\}]$$

$$\text{implies } \beta \in \text{int } \cap\{[\![S^1_u((Seq(u) \wedge Seq(u)) \rightarrow A(w*u*2^{s+1}))]\!]^T_{A^{w,u,s}_{w,u,s}} : s \in N_I\}.$$

Since $[\![Seq(1)]\!]^T_A = N^N$ for each A and $w*1 = w$, we have ③ implies

$$[\beta \notin [\![(\exists z \leq lh(w))(Seq(w) \rightarrow R(\lambda t\overline{t(w)_t \dot{-} 1}(z))]\!]^T_{A^{w,s}_{w,s}}$$

$$\text{and } \beta \in \cap\{[\![Seq(w) \rightarrow A(w*2^{s+1})]\!]^T_{A^{w,s}_{w,s}} : s \in N_I\}]$$

$$\text{implies } \beta \in \text{int}\cap\{[\![Seq(w) \rightarrow A(w*2^{s+1})]\!]^T_{A^{w,s}_{w,s}} : s \in N_I\}.$$

[The reduction to $A^{w,s}_{w,s}$ is possible, since 1 is substitutable for u in each of the formulas B and then $S^1_u(B)$ does not contain u.]

But the antecedents of this are the part ② of our initial assumption and our assumption ③. Hence we have

$$\beta \in \text{int } \cap\{[\![Seq(w) \rightarrow A(w*2^{s+1})]\!]^T_{A^{w,s}_{w,s}} : s \in N_I\},$$

i.e. $\beta \in [\![\forall s(Seq(w) \rightarrow A(w*2^{s+1}))]\!]^T_{A^w_w}$.

Since s does not appear in $Seq(w)$ we have

$$\beta \in [\![Seq(w) \rightarrow \forall s A(w*2^{s+1})]\!]^T_{A^w_w},$$

which together with $\beta \in [\![\forall w(Seq(w) \rightarrow (\forall s A(w*2^{s+1}) \rightarrow Aw))]\!]^T_A$ implies $\beta \in [\![Seq(w) \rightarrow Aw]\!]^T_{A^w_w}$ for this, assumed $w \in N_I$.

But this contradicts part ① of our initial assumption.

Hence for this assumed $w \in N_I$,

$\beta \notin \cap\{[\![Seq(w) \rightarrow A(w*2^{s+1})]\!]^T_{A^{w,s}_{w,s}} : s \in N_I\}$, i.e. for this

$w \in N_I$, there is an $s_1 \in N_I$ such that

$$\beta \notin [\![Seq(w) \rightarrow A(w*2^{s_1+1})]\!]^T_{A^{w,s_1}_{w,s_1}}.$$

Using $\beta \in [\![\forall w(\mathrm{Seq}(w) \to (Rw \to Aw))]\!]^{T}_{A}$ and the definition of $[\![A \to B]\!]^{T}_{A}$, we have for this $w \in N_I$ that there is an $s_1 \in N_I$ such that $\beta \notin [\![\mathrm{Seq}(w) \to R(w*2^{s_1+1})]\!]^{T}_{A^{w,s_1}_{w,s_1}}$.

I.e. for this $w \in N_I$, there is an $s_1 \in N_I$ such that

$\beta \notin [\![\mathrm{Seq}(w) \to R(\lambda t\,\overline{(w*2^{s_1+1})_t \dot{-} 1}(\mathrm{lh}(w*2^{s_1+1})))]\!]^{T}_{A^{w,s_1}_{w,s_1}}$.

Since $\mathrm{lh}(w*2^{s_1+1})$ is substitutable for z in

$\mathrm{Seq}(w) \to R(\lambda t\,\overline{(w*2^{s_1+1})_t \dot{-} 1}(z))$, using 4.3.5 and

$\mathrm{lh}(w*2^{s_1+1}) = \lambda\gamma.[\![\mathrm{lh}(w*2^{s_1+1})]\!]_{A^{w,s_1}_{w,s_1},\beta}$, we have for this

$w \in N_I$ that there is an $s_1 \in N_I$ such that

$\beta \notin [\![\mathrm{Seq}(w) \to R(\lambda t\,\overline{(w*2^{s_1+1})_t \dot{-} 1}(z))]\!]^{T}_{A^{w,\mathrm{lh}(w*2^{s_1+1})}_{w,z}}$ — (5)

Using the easily established result

$\beta \notin [\![(\exists z \leqslant \mathrm{lh}(w))B(w,z)]\!]^{T}_{A^{w}_{w}}$ iff, for each $z \leqslant \mathrm{lh}(w)$,

$\beta \notin [\![B(w,t)]\!]^{T}_{A^{w,z}_{w,z}}$ [where $z \leqslant \mathrm{lh}(w)$ is an abbreviation of "for some x, for each γ, $x(\gamma) + z(\gamma) = \mathrm{lh}(w(\gamma))$" where $+$, lh are the primitive recursive functions on N] the part (2) of the initial assumption becomes for each $z \leqslant \mathrm{lh}(w)$,

$\beta \notin [\![\mathrm{Seq}(w) \to R(\lambda t\,\overline{(w)_t \dot{-} 1}(z))]\!]^{T}_{A^{w,z}_{w,z}}$.

Since for $z \leqslant \mathrm{lh}(w)$, $\lambda\overline{t\,(w)_t \dot{-} 1}(z) = \lambda t\,\overline{(w*2^{s_1+1})_t \dot{-} 1}(z)$,

we have for each $z \leqslant \mathrm{lh}(w)$

$\beta \notin [\![\mathrm{Seq}(w) \to R(\lambda t\,\overline{(w*2^{s_1+1})_t \dot{-} 1}(z))]\!]_{A^{w,z,s_1}_{w,z,s_1}}$.

The extension to A^{w,z,s_1}_{w,z,s_1} is allowed since s_1, does not occur in $\mathrm{Seq}(w) \to R(\lambda\overline{t\,(w)_t \dot{-} 1}(z))$].

Combining this with (5) we have, for each $z \leqslant lh(w*2^{s_1+1})$,

$$\beta \notin [\![Seq(w) \to R(\overline{\lambda t(w*2^{s_1+1})_t} \dot{-} 1(z))]\!]^{T\,w,z,s_1}_{A_{w,z,s_1}} ,$$

i.e. $\beta \notin [\![(\exists z \leqslant lh(w*2^{s_1+1}))(Seq(w) \to R(\overline{\lambda t(w*2^{s_1+1})_t \dot{-} 1(z)}))]\!]^{T\,w,s_1}_{A_{w,s_1}}$.

Since $[\![Seq(2^{s_1+1})]\!]^{T}_{A} = N^N$ for each A and $A^{s_1}_{s_1} = A^{2^{s_1+1}}_{2^{s_1+1}}$, we have for this $w \in N_I$, there is an $s_1 \in N_I$ such that

$$\beta \notin [\![(\exists z \leqslant lh(w*2^{s_1+1}))((Seq(w) \wedge Seq(2^{s_1+1}))$$
$$\to R(\overline{\lambda t(w*2^{s_1+1})_t} \dot{-} 1(z)))]\!]^{T\,w,2^{s_1+1}}_{A_{w,2^{s_1+1}}} , \qquad (6)$$

If we suppose

$$\beta \in \cap\{ [\![(Seq(w) \wedge Seq(2^{s_1+1})) \to A(w*2^{s_1+1})]\!]^{T\,w,2^{s_1+1},s}_{A_{w,2^{s_1+1},s}} : s \in N_I\},$$

then this and (6) are the antecedents of (3) applied to $u = 2^{s_1+1}$, so that using the argument above applied to this case we get a contradiction. Hence for this $w \in N_I$, there are $s_1, s_2 \in N_I$ such that

$$\beta \notin [\![(Seq(w) \wedge Seq(2^{s_1+1})) \to A(w*2^{s_1+1}*2^{s_1+1})]\!]^{T\,w,s_1,s_2}_{A_{w,s_1,s_2}} .$$

Hence using the argument above applied to this case, we get

$$\beta \notin [\![(Seq(w) \to R(w*2^{s_1+1}*2^{s_2+1})]\!]^{T\,w,s_2,s_2}_{A_{w,s_1,s_2}},$$

and so deduce as before

that for this $w \in N_I$, there are $s_1, s_2 \in N_I$ such that

$$\beta \notin [\![(\exists z \leqslant lh(w*2^{s_1+1}*2^{s_2+1}))$$
$$(Seq(w) \to R(\lambda t(\overline{w*2^{s_1+1}*2^{s_2+1})_t \dot{-} 1(z)}))]\!]^{T\,w,2^{s_1+1},2^{s_2+1}}_{A_{w,2^{s_1+1},2^{s_2+1}}} ,$$

etc.

So in general we have: for this $w \in N_I$, there are $s_1, s_2, \ldots \in N_I$ such that for each $x \in N_I$,

$$3 \notin [\![\mathbf{Seq(w)} \to \mathbf{R}(\lambda t\, (w * 2^{s_1 +1} * \ldots * 2^{s_{1+(t \dot{-} lh(w))}+1})_t \dot{-} 1(w))]\!]^{T}_{A^{w,s_1,s_2,\ldots,x}_{w,s_1,s_2,\ldots,x}} ,$$

where $A^{w,s_1,s_2,\ldots,x}_{w,s_1,s_2,\ldots,x}(s_{1+(t \dot{-} lh(w))})(\gamma)$

$$= s_{[\![1+(t \dot{-} lh(w))]\!]_{A^{w,x,t}_{w,x,t}}, \gamma} ,$$

i.e. $s_{(1+(t \dot{-} lh(w)))}(\gamma)$.

Since $\lambda t\, (w * 2^{s_1 +1} * \ldots * 2^{s_{1+(t \dot{-} lh(w))}+1})_t \dot{-} 1$ is substitutable for α in $\mathbf{Seq(w)} \to \mathbf{R}\bar{\alpha}x$, using 4.3.6 and

$$[\![\lambda t\, .(w * 2^{s_1 +1} * \ldots * 2^{s_{1+(t \dot{-} lh(w))}+1})_t \dot{-} 1]\!]^{T}_{A^{w,s_1,s_2,\ldots x}_{w,s_1,s_2\ldots,x}}, \gamma$$

$$= \lambda t(\gamma).((w * 2^{s_1+1} * \ldots * 2^{s_{1+(t \dot{-} lh(w))}+1})_t \dot{-} 1)(\gamma),$$

$$= \lambda t(\gamma).(w(\gamma) * 2^{s_1+1} * \ldots * 2^{s_{(1+(t \dot{-} lh(w)))}(\gamma)+1})_{t(\gamma)} \dot{-} 1$$

we have, for this $w \in N_I$, these $s_1, s_2, \ldots \in N_I$, for each $x \in N_I$,

$$\beta \notin [\![\mathbf{Seq(w)} \to \mathbf{R}(\bar{\alpha}x)]\!]^{T}_{A^{w,s_1,\ldots,x,}_{w,s_1,\ldots,x,\alpha}} \lambda t(\gamma)(w(\gamma) * 2^{s_1+1} * \ldots * 2^{s_{(1+(t \dot{-} lh(w)))}+1})_{t(\gamma)} \dot{-} 1 \qquad - \; 7$$

Now if we define $\Phi_0 : N^N \to N^N$ by, for each $\gamma \in N^N$, $t \in N$,

$$\Phi_0(\gamma)(t) = (w(\gamma))_t \dot{-} 1 \qquad , t < lh(w(\gamma)),$$

$$= s_{(\lambda\gamma.1+(\lambda\gamma.t \dot{-} lh(w))(\gamma)}(\gamma), \; t \geqslant lh(w(\gamma)),$$

which is constant wrt γ since $w, s_{(\lambda\gamma.1+(\lambda\gamma.t \dot{-} lh(w)))(\gamma)}$ are constant wrt γ. Then $\Phi_0 \in C(N^N)$ and 7 imply for this $w \in N_I$, there are $s_1, s_2, \ldots \in N_I$, for some $\Phi_0 \in C(N^N)$, for each $x \in N_I$, such that

$$\beta \notin [\![\mathrm{Seq}(w) \to R(\bar{\alpha}x)]\!]^{T}_{A^{w,s_1,\ldots,x,\Phi_0}_{w,s_1,\ldots,x,\alpha}},$$

i.e. $\beta \notin [\![R(\bar{\alpha}x)]\!]^{T}_{A^{x,\Phi_0}_{x,\alpha}}$

[since $w, s_1, \ldots$ do not appear in $R(\bar{\alpha}x)$ and by definition of $[\![A \to B]\!]^{T}_{A}$].

This contradicts $\beta \in [\![\forall\alpha \exists! x R(\bar{\alpha}x)]\!]^{T}_{A}$, since this implies for each $\Phi \in C(N^N)$, there is some $x \in N_I$ such that $\beta \in [\![R(\bar{\alpha}x)]\!]^{T}_{A^{x,\Phi}_{x,\alpha}}$.

4.3.16. Remark. 4.3.15 could be weakened by changing $\exists z \leqslant lh(w)$ to $\exists z < lh(w)$. However, since $\beta \in [\![\forall w(\mathrm{Seq}(w) \to (Rw \to Aw))]\!]^{T}_{A}$ and $\beta \notin [\![\mathrm{Seq}(w) \to Aw]\!]^{T}_{A^{w}_{w}}$ imply $\beta \notin [\![\mathrm{Seq}(w) \to Rw]\!]^{T}_{A^{w}_{w}}$, and $w = \lambda \overline{t(w)_t \dot{-} 1}(lh(w))$, we see that the lemma with the weakened $\exists z < lh(w)$ is equivalent to 4.3.15, but needs some extra steps in the proof. Thus it is for convenience that we chose the stronger condition $\exists z \leqslant lh(w)$.

4.3.17. Proposition. The Axiom schema $\mathbf{BI!}$ is valid in N_I.

Proof. By contradiction. For some A, suppose $\beta_0 \in N^N$ and V_0 is a neighbourhood of β_0 with

$$V_0 \subseteq [\![\forall\alpha\exists! x R(\bar{\alpha}x) \wedge \forall w(\mathrm{Seq}(w) \to (Rw \to Aw)) \wedge \forall w(\mathrm{Seq}(w) \to (\forall s A(w * 2^{s+1}) \to Aw))]\!]^{T}_{A} \text{ and } \beta_0 \notin [\![A(1)]\!]^{T}_{A}.$$

Since 1 is substitutable for w in Aw and $[\![\mathrm{Seq}(1)]\!]^{T}_{A} = N^N$ for each A, 4.3.5 implies

$$\beta_0 \notin [\![\mathrm{Seq}(w) \to Aw]\!]^{T}_{A^{1}_{w}}, \qquad - (1)$$

Using $\beta_0 \in [\![\forall w(\mathrm{Seq}(w) \to (Rw \to Aw))]\!]^{T}_{A}$ (1) implies

$\beta_0 \notin \| Seq(w) \rightarrow Rw \|^{T}_{A_w^1}$, i.e. using 4.3.5,

$\beta_0 \notin [\![Seq(1) \rightarrow R(1)]\!]^{T}_{A}$. Using the equivalence

(*) $\beta_0 \notin [\![(\exists z \leqslant lh(w)) B(w,z)]\!]^{T}_{A_w} w$ iff, for each $z \leqslant lh(w)$,

$\beta_0 \notin [\![B(w,t)]\!]^{T}_{A_{w,z}} w,z$, and $1 = \overline{\lambda t(1)_t \dot{-} 1(z)}$ for each z, we have

for each $z \leqslant 0$, $\beta_0 \notin [\![Seq(w) \rightarrow R(\lambda t(w)_t \dot{-} 1(z))]\!]^{T}_{A_{w,z}} w,z$;

i.e. $\beta_0 \notin [\![(\exists z \leqslant lh(w))(Seq(w) \rightarrow R(\lambda \overline{t(w)_t \dot{-} 1(z)}))]\!]^{T}_{A_{w,z}} 1,z$ – (2)

(1) and (2) are the hypotheses of 4.3.15 restricted to $w = 1$, so that by 4.3.15, using 4.3.5 on w and the equivalence (*) we have, there is some $u_1 \in N_I$ such that

$$\left.\begin{array}{l} \beta_0 \notin [\![(\exists z \leqslant lh(u_1))(Seq(u_1) \rightarrow R(\lambda t.(u_1)_t \dot{-} 1)(z))]\!]^{T}_{A_{u_1}} u_1 \\ \text{and } \beta_0 \in \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1,s : s \in N_I\} \\ \text{and } \beta_0 \notin \operatorname{int} \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1,s : s \in N_I\}. \end{array}\right\} \quad - (3)$$

Since $V_0 \subseteq [\![\forall\alpha\exists! x R(\bar{\alpha}x)]\!]^{T}_{A}$ by assumption, we have $V_0 \subseteq [\![\exists! x R(\lambda \overline{t(u_1)_t \dot{-} 1(x)})]\!]^{T}_{A_{u_1}} u_1$.

So by 4.3.14, for each $z \in N_I$, $[\![R(\overline{\lambda t(u_1)_t \dot{-} 1(z)})]\!]^{T}_{A_{u_1,z}} u_1,z$

is clopen in V_0. So restricting ourselves to the $z \leqslant lh(u_1)$, using the equivalence (*), $[\![Seq(u_1)]\!]_{A_{u_1}} u_1$ is clopen, and 4.3.5, we have

$$[\![(\exists z \leqslant lh(u_1))(Seq(u_1) \rightarrow R(\lambda \overline{t(u_1)_t \dot{-} 1(t)}))]\!]^{T}_{A_{u_1}} u_1$$

is clopen in V_0.

Therefore there is a neighbourhood V_1 of β_0, namely $V_0 \cap - [\![(\exists z \leqslant lh(u_1))(Seq(u_1) \rightarrow R(\lambda \overline{t(u_1)_t \dot{-} 1(z)}))]\!]^{T}_{A_u} u$, such that

$V_1 \cap [\![(\exists z \leqslant lh(u_1))(Seq(u_1) \rightarrow R(\overline{\lambda t(u_1)_t \dot{-} 1(z)}))]\!]^{T}_{A_{u_1}} u_1 = \phi$.

Suppose for each $\beta \in V_1$,

$\beta \in \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1, s \; : \; s \in N_I\}$.

Then since int $V_1 = V_1$ this implies

$\beta \in \text{int} \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1, s \; : \; s \in N_I\}$.

This contradicts $\beta_0 \in V_1$ and

$\beta_0 \notin \text{int} \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1, s \; : \; s \in N_I\}$ by 3.

Therefore since $\beta_0 \in \cap\{[\![Seq(u_1) \rightarrow A(u_1 * 2^{s+1})]\!]^{T}_{A_{u_1,s}} u_1, s : s \in N_I\}$,

there exist $\beta_1 \neq \beta_0 \in N^N$, $s_1 \in N$ such that

(a_1) $\beta_1 \in V_1 \subset V_0$ (by definition of V_1),

(b_1) $\beta_1 \notin [\![Seq(u_1) \rightarrow A(u_1 * 2^{s_1+1})]\!]^{T}_{A_{u_1,s_1}} u_1, s_1$ (by above),

(c_1) $\bar{\beta}_1(1) = \bar{\beta}_0(1)$.

Using $V_0 \subseteq [\![\forall w(Seq(w) \rightarrow (Rw \rightarrow Aw))]\!]^{T}_{A}$ and the equivalence $*$,

(a_1) and (b_1) imply

$\beta_1 \notin [\![(\exists z \leqslant lh(u * 2^{s_1+1}))(Seq(u_1) \rightarrow \overline{R(\lambda t.(u_1 * 2^{s_1+1}) \dot{-} 1(z))})]\!]^{T}_{A_{u_1,s_1}} u_1, s_1$.

This and (b_1) are the hypotheses of 4.3.15 restricted to $w = u_1 * 2^{s_1+1}$, so we can use 4.3.15 again.

In general we show by 4.3.15 and the above argument that there is a nbd $V_{n+1} \subseteq V_n \subseteq \ldots \subseteq V_0$ of β_n, such that

$V_{n+1} \cap [\![(\exists z \leqslant lh(u_1 * 2^{s_1+1} * \ldots * u_{n+1}))$

$((Seq(u_1) \wedge \ldots \wedge Seq(u_{n+1})) \rightarrow$

$\overline{R(\lambda t(u_1 * 2^{s_1+1} * \ldots * u_{n+1})_t \dot{-} 1(z))})]\!]^{T}_{A_{u_1 \ldots}} u_1 \ldots$,

so that there is a $\beta_{n+1} \neq \beta_n \in N^N$, $s_{n+1} \in N_I$, such that

$(a_{n+1})\ \beta_{n+1} \in V_{n+1} \subseteq V_n$,

(b_{n+1}) $\beta_{n+1} \notin [\![(\mathrm{Seq}(u_1) \wedge \ldots \wedge \mathrm{Seq}(u_{n+1})) \to$

$\mathrm{A}(u_1 * 2^{s_1+1} * \ldots * u_{n+1} * 2^{s_{n+1}+1})]\!]^{T}_{A\, u_1, s_1, \ldots, u_{n+1}, s_{n+1}}{}^{u_1, s_1, \ldots, u_{n+1}, s_{n+1}},$

(c_{n+1}) $\bar{\beta}_{n+1}(n+1) = \bar{\beta}_n(n+1)$.

As before (a_{n+1}) and (b_{n+1}) imply

$$\beta_{n+1} \notin [\![(\exists z \leq \mathrm{lh}(u_1 * 2^{s_1+1} * \ldots *_{n+1} * 2^{s_{n+1}+1}))$$
$$((\mathrm{Seq}(u_1) \wedge \ldots \wedge \mathrm{Seq}(u_{n+1})) \to$$
$$\overline{\mathrm{R}(\lambda t(u_1 * 2^{s_1+1} * \ldots * u_{n+1} * 2^{s_n+1})_t \dot{-} 1(z)}))]\!]^{T}_{A\, u_1, s_1 \ldots}{}^{u_1, s_1 \ldots} \qquad (4)$$

The conditions (c_n) imply $\beta_0, \beta_1, \ldots$ converge to some $\beta \in N^N$. Since V_0 is clopen the conditions (a_n) imply $\beta \in V_0$.

Hence $\beta \in [\![\forall\alpha\exists! x \mathrm{R}(\bar{\alpha}x)]\!]^{T}_{A}$ - (5) .

Consider $\Phi \in N^N \to N^N$ given by $\lambda\gamma.\lambda n(\gamma).(w_{(n+1)(\gamma)})_{n(\gamma)} \dot{-} 1(\gamma)$, where w_{n+1} is $u_1 * 2^{s_1+1} * \ldots * u_{n+1} * 2^{s_{n+1}+1}$. Φ is constant wrt γ so is an element of $C(N^N)$. Moreover

$$\Phi = \lambda\gamma.[\![\lambda n.(w_{n+1})_n \dot{-} 1]\!]_{A\, u_1, s_1, \ldots, u_{n+1}, s_{n+1}}{}^{u_1, s_1, \ldots, u_{n+1}, s_{n+1}, \gamma},$$

where $w_{n+1} = u * 2^{s_1+1} * \ldots * u_{n+1} * 2^{s_{n+1}+1}$.

Since $\lambda n.(w_{n+1})_n \dot{-} 1$ is substitutable for α in $\exists! x \mathrm{R}(\bar{\alpha}x)$ by 4.3.6 and (5),

$$\beta \in [\![\exists! x \mathrm{R}(\overline{\lambda n.(w_{n+1})_n \dot{-} 1}(x))]\!]^{T}_{A\, n_1, s_1, \ldots, u_{n+1}, s_{n+1}}{}^{u_1, s_1, \ldots, u_{n+1}, s_{n+1}}.$$

Therefore for some $t \in N_I$ and neighbourhood U,

$$\beta \in U \subseteq [\![\mathrm{R}(\overline{\lambda n(w_{n+1})_n \dot{-} 1}(x))]\!]^{T}_{A\, u_1, s_1, \ldots, u_{n+1}, s_{n+1}, x}{}^{u_1, s_1, \ldots, u_{n+1}, s_{n+1}, t}.$$

Using 4.3.5 and $t = \lambda\gamma.\llbracket t \rrbracket \mathcal{A}^{u_1,s_1,\ldots,u_{n+1},s_{n+1}}_{u_1,s_1,\ldots,u_{n+1},s_{n+1}},\beta$, and

that t is a closed term and so substitutable for x in

$R(\lambda n \overline{(w_{n+1})_n \dot{-} 1}(x))$, we have

$$\beta \in U \subseteq \llbracket R(\lambda n \overline{(w_{n+1})_n \dot{-} 1}(t)) \rrbracket^T \mathcal{A}^{u_1,s_1,\ldots,u_{n+1},s_{n+1}}_{u_1,s_1,\ldots,u_{n+1},s_{n+1}}.$$

So that by the definition of $\llbracket A \to B \rrbracket^T_A$ we have

$$\beta \in U \subseteq \llbracket (Seq(u_1) \wedge \ldots \wedge Seq(u_{n+1})) \to R(\lambda n \overline{(w_{n+1})_n \dot{-} 1}(t)) \rrbracket^T \mathcal{A}^{u_1,s_1,\ldots,u_{n+1},s_{n+1}}_{u_1,s_1,\ldots,u_{n+1},s_{n+1}}.$$

Therefore for some k, for each $m \geqslant k$,

$$\beta_m \in U \subseteq \llbracket (Seq(u_1) \wedge \ldots \wedge Seq(u_{n+1})) \to R(\lambda n \overline{(w_{n+1})_n \dot{-} 1}(t)) \rrbracket^T \mathcal{A}^{u_1,s_1,\ldots,u_{n+1},s_{n+1}}_{u_1,s_1,\ldots,u_{n+1},s_{n+1}}.$$

Choose $m > k$ and $m > t$, where

$$\lambda\gamma.t = \lambda\gamma.\llbracket t \rrbracket \mathcal{A}^{u_1,s_1,\ldots,u_{n+1},s_{n+1}}_{u_1,s_1,\ldots,u_{n+1},s_{n+1}},\beta;$$

then $w_m = w_{t+1} * u_{t+2} * 2^{s_{t+2}+1} * .. \; u_m * 2^{s_m+1}$

and so $\lambda n \overline{(w_m)_n \dot{-} 1}(t) = \lambda n \overline{(w_{n+1})_n \dot{-} 1}(t)$.

Hence $\beta_m \in \llbracket (Seq(u_1) \wedge \ldots \wedge Seq(u_m)) \to R(\lambda n \overline{(w_m)_n \dot{-} 1}(t)) \rrbracket^T \mathcal{A}^{u_1,s_1,\ldots,u_m,s_m}_{u_1,s_1,\ldots,u_m,s_m}$

[we are able to detach the $Seq(u_{m+1}),\ldots,Seq(u_{n+1})$ by the definition of $\llbracket A \to B \rrbracket^T_A$].

Since $\lambda\gamma.t \leqslant lh(w_n)$, using the equivalence (*) on (4) for $n+1 = m$, we obtain a contradiction.

<u>4.3.18. Remark</u>. To show **BC-N!** is valid in N_I we need the following lemma.

<u>4.3.19. Lemma.</u> Suppose $A(\alpha,x)$ is a formula such that for each A, for each $\beta,\ \delta \in N^N$, there exist z, b $\in N$ such that

for each $\gamma \in N^N$, each $\Phi \in C(N^N)$, if $\bar{\gamma}z = \bar{\beta}z$ and $[\overline{\Phi(\gamma)}](z) = \bar{\delta}z$ then $\gamma \in [\![(A\alpha,x)]\!]^{T}_{A_{\alpha,x}}\Phi, \lambda\gamma.b$.

Then there exists a $\Psi \in C(N^N)$ such that

$[\![\forall\alpha\exists y(\tau(\bar{\alpha}y) > 0 \wedge \forall x(\tau(\bar{\alpha}x > 0 \rightarrow x = y) \wedge A(\alpha,\tau(\bar{\alpha}y) \dot{-} 1))]\!]^{T}_{A_\tau}\Psi = N^N$.

<u>Proof.</u> Define $B(\beta,n,y)$ by:

$B(\beta,n,y) =_{df}$ Seq(n) & $(\gamma \in N^N)(\Phi \in C(N^N))$

$[\bar{\gamma}(lhn) = \bar{\beta}(lhn)$ & $[\overline{\Phi(\gamma)}](lhn) = n \Rightarrow \gamma \in [\![A(\alpha,x)]\!]^{T}_{A_{\alpha,x}}\Phi, \lambda\gamma.y]$.

Define Ψ by: for each $\beta \in N^N$, $n \in N$,

$\Psi(\beta)(n) = b+1$ iff $B(\beta,n,b)$ & $(y < b) \sim B(\beta,n,y)$ &

$(z < lhn)(y) \sim B(\beta, \Pi_{i<z} P_i^{(n)i}, y)$,

$= 0$ iff $(y) \sim B(\beta,n,y)$ or $(Ey < b)B(\beta,n,y)$ or

$(Ez < lhn)(Ey)B(\beta,\Pi_{i<z} P_i^{(n)i}, y)$;

i.e. $\Psi(\beta)(n) = b+1$ iff $B(\beta,n,b)$, where b is the least y such that $B(\beta,n,y)$ and n is the least seq(m) such that $B(\beta,m,b)$.

Hence Ψ is a function.

Moreover $\Psi \in C(N^N)$ for

given α, given n a sequence,

$\Psi(\alpha) \in \{n\}$ implies $\overline{\Psi\alpha}(lhn) = n$

implies $\Psi(\alpha)(i) = (n)_i^-$ for each $i < lhn$, where

$(n)_i > 0$.

Let $j = \max_{i<lhn} \{lhi\}$, then $\alpha \in \{\bar{\alpha}j\}$ and for each β, $\beta \in \{\bar{\alpha}j\}$

implies for each $i < lhn$, $\bar{\alpha}(lhi) = \bar{\beta}(lhi)$

implies for each $i < lhn$, $B(\alpha,i,((n)^-_i)^-) = B(\beta,i,((n)^-_i)^-)$

[by definition of $B(\beta,n,y)$]

implies for each $i < lhn$, $\Psi(\alpha)(i) = (n)^-_i$ iff $\Psi(\beta)(i) = (n)^-_i$

[by definition of Ψ]

implies for each $i < lhn$, $\Psi(\beta)(i) = (n)^-_i$

[since for each $i < lhn$,

$\Psi(\alpha)(i) = (n)^-_i$]

implies $\overline{\Psi\beta}(lhn) = n$

implies $\Psi\beta \in \{n\}$.

Since $Seq(\bar{\delta}z)$, by assumption $(\beta)(\delta)(Ez)(Eb)B(\beta,\bar{\delta}z,b)$, so that $(\beta)(\Phi(\beta))(Ey(\beta))(Eb)B(\beta,[\overline{\Phi(\beta)}]y(\beta),b)$. Now for each $\beta,\Phi(\beta)$ we can choose the least b and $y(\beta)$ such that $B(\beta,[\overline{\Phi(\beta)}]y(\beta),b)$.

Hence for this b and $y(\beta)$, $\Psi(\beta)([\overline{\Phi(\beta)}]y(\beta)) = b+1 \neq 0$ and so $\cup\{\{\beta : \Psi(\beta)([\overline{\Phi(\beta)}]y(\beta)) \neq 0\} : y \in N_I\} = N^N$.

Hence $\cup\{[\![\tau(\bar{\alpha}\mathbf{y}) > \mathbf{0}]\!]^{T}_{A}{}^{\Phi,\Psi,y}_{\alpha,\tau,\mathbf{y}} : y \in N_I\} = N^N$. $\quad$ - 1

For each $y \in N_I$, since $[\![\mathbf{x} = \mathbf{y}]\!]^{T}_{A}{}^{x,y}_{\mathbf{x},\mathbf{y}} = N^N$ iff $x = y$, and if $x \neq y$ such that $y(\beta)$ is the least n with $B(\beta,\overline{\Phi(\beta)}n,b)$ then $[\![\tau(\bar{\alpha}\mathbf{x}) > \mathbf{0}]\!]^{T}_{A}{}^{\Phi,\Psi,x}_{\alpha,\tau,\mathbf{x}} = \phi$

$$\cap\{[\![\tau(\bar{\alpha}\mathbf{x}) > \mathbf{0} \rightarrow \mathbf{x} = \mathbf{y}]\!]^{T}_{A}{}^{\Phi,\Psi,y,x}_{\alpha,\tau,\mathbf{y},\mathbf{x}} : x \in N_I\} = N^N.$$

Hence $\cap\, [\![\forall\mathbf{x}(\tau(\bar{\alpha}\mathbf{x}) > \mathbf{0} \rightarrow \mathbf{x} = \mathbf{y})]\!]^{T}_{A}{}^{\Phi,\Psi,y}_{\alpha,\tau,\mathbf{y}} : y \in N_I\} = N^N$ $\quad$ - 2

For each $\beta \in N^N$, each $\Phi(\beta)$, there is some $y \in N_I$ such that $B(\beta,[\overline{\Phi(\beta)}]y(\beta),\Psi(\beta)[\overline{\Phi(\beta)}]y(\beta) \dot{-} 1)$ by our choice of Ψ.

Hence $\beta \in [\![\mathbf{A}(\alpha\ \mathbf{x})]\!]^{T}_{A}{}^{\Phi,\lambda\gamma.\Psi(\beta)[\overline{\Phi(\beta)}]y(\beta) \dot{-} 1}_{\alpha,\ \mathbf{x}}$ by definition of $B(\beta,n,y)$,

since $[\overline{\Phi(\beta)}](lh[\overline{\Phi(\beta)}]y(\beta)) = [\overline{\Phi(\beta)}]y(\beta)$.

Therefore for each Φ, each β, there is some $y \in N_I$ such that

$$\beta \in [\![A(\alpha,x)]\!]^{T}_{A\,\alpha,\ x}{}^{\Phi,\lambda\gamma.[\![\tau(\bar{\alpha}y) \dot{-} 1]\!]_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y}} ,\beta ,$$

so that using 4.3.5,

$$\beta \in [\![A(\alpha,\tau(\bar{\alpha}y) \dot{-} 1)]\!]^{T}_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y}$$

Hence $\cup\{[\![A(\alpha,\tau(\bar{\alpha}y) \dot{-} 1)]\!]^{T}_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y} : y \in N_I\} = N^N$ — ③

①, ②, ③ imply $\cup\{[\![\tau(\bar{\alpha}y) > 0]\!]^{T}_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y} \cap$

$[\![\forall x(\tau(\bar{\alpha}x) > 0 \rightarrow x = y)]\!]^{T}_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y} \cap$

$[\![A(\alpha,\tau(\bar{\alpha}y) \dot{-} 1)]\!]^{T}_{A\,\alpha,\tau,y}{}^{\Phi,\Psi,y} : y \in N_I\} = N^N$.

Since ①, ②, ③ hold for each Φ we have

$$[\![\forall\alpha\exists y(\tau(\bar{\alpha}y) > 0) \wedge \forall x(\tau(\bar{\alpha}x) > 0 \rightarrow x = y) \wedge A(\alpha,\tau(\bar{\alpha}y) \dot{-} 1))]\!]^{T}_{A_\tau}{}^{\Psi} = N^N .$$

<u>4.3.20. Proposition</u>. The Axiom schema **BC − N!** is valid in N_I.

<u>Proof</u>. Suppose $\beta_0 \in N^N$ and V_0 is a neighbourhood of β_0 such that $V_0 \subseteq [\![\forall\alpha\exists! y A(\alpha,y)]\!]^{T}_{A}$. Using 4.3.14 we have, for each $\Phi \in C(N^N)$ and each $\lambda\gamma.y \in N_I$, the set $[\![A(\alpha,y)]\!]_{A\,\alpha,\ y}{}^{\Phi,\lambda\gamma.y}$ is clopen relative to V_0. Moreover, for each $\Phi \in C(N^N)$, the sets $[\![A(\alpha,y)]\!]^{T}_{A\,\alpha,\ y}{}^{\Phi,\lambda\gamma y} \cap V_0$ for $y = 0,1,\ldots$ form a partition of V_0, evaluating the quantifiers $\forall\alpha$ and $\exists! y$ so that these sets are disjoint.

Let $\Phi \in C(N^N)$, suppose $\gamma \in V$ and $\Phi(\gamma) = \alpha$. Define Φ_α by: for each δ, $\Phi_\alpha(\delta) = \alpha$. Clearly $\Phi_\alpha \in C(N^N)$ and $\gamma \in \{\beta : \Phi_\alpha(\beta) = \Phi(\beta)\}$. Hence by 4.3.11, $\gamma \in [\![A(\alpha,y)]\!]^{T}_{A\,\alpha\ ,y}{}^{\Phi\alpha,\lambda\gamma.y}$.

Define $F : N^N \times N^N \to N$ by

$$F(\gamma,\alpha) =_{df} y \quad \text{iff } \gamma \in V_0 \text{ and } \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Phi\alpha,\lambda\gamma.y},$$

$$=_{df} 0 \quad \text{iff } \gamma \notin V_0.$$

Since V_0 is clopen, if F is continuous on $V_0 \times N^N$ then it is continuous on $N^N \times N^N$. Suppose F is not continuous on $V_0 \times N^N$. Then there exist distinct points $\gamma^*, \gamma_1, \gamma_2, \ldots \in V_0$ and distinct points $\alpha^*, \alpha_1, \alpha_2, \ldots \in N^N$ such that $\{\gamma_n\} \to \gamma^*$, $\{\alpha_n\} \to \alpha^*$, and by choosing a suitable subsequence if necessary, $F(\gamma_n, \alpha_n) \neq F(\gamma^*, \alpha^*)$. Note that N as usual has the discrete topology.

Choose some $\Psi \in C(N^N)$ such that $\Psi(\gamma_n) = \alpha_n$ for each n, and $\Psi(\gamma^*) = \alpha^*$. Let $y^* = F(\gamma^*, \alpha^*)$, then by 4.3.11, $\gamma^* \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Psi,\lambda\gamma.y^*}$. Hence for n sufficiently large, $\gamma_n \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Psi,\lambda\gamma.y^*}$.

However $\gamma_n \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Psi,\lambda\gamma.y_n}$ for each n, where $y_n = F(\gamma_n, \alpha_n)$. Since $y_n \neq y^*$ for each n,

$$[\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Psi,\lambda\gamma.y_n} \cap [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Psi,\lambda\gamma.b} \cap V_0 = \phi.$$

A contradiction, hence F is continuous.

F continuous implies for each y, $F^{-1}[y]$ is open

implies for each y, $F^{-1}[y] \cap V_0 \times N^N$ is open

implies for each y,

$$\{(\gamma,\alpha) : \gamma \in V_0 \ \&\ \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Phi\alpha,\lambda\gamma.y}\}$$

is open.

By 4.3.11, since $\alpha = \Phi_\alpha(\gamma) = \Phi(\gamma)$, this

implies for each y,

$$\{(\gamma,\Phi(\gamma)) : \gamma \in V_0 \& \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A}{}_{\alpha,\mathbf{y}}^{\Phi,\lambda\gamma.y}\}$$

is open.

By definition of product topology this implies for each y, there exist neighbourhoods $\{m\}$, $\{n\}$ such that

(1) − $(\gamma \in V_0)(\Phi)(\gamma \in \{m\} \;\&\; \Phi(\gamma) \in \{n\} \Rightarrow \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A_{\alpha,\mathbf{y}}} \Phi, \lambda\gamma . y)$.

Given $\beta \in V_0$, for each Ψ, there is exactly one $y_\Psi \in N$ such that

$\beta \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A_{\alpha,\mathbf{y}}} \Psi, \lambda\gamma . y_\Psi$, as shown in the first paragraph.

By (1) there are neighbourhoods $\{m_\Psi\}$, $\{n_\Psi\}$ of β and $\Phi(\beta)$ such that

$$(\gamma \in V_0)(\Phi)(\gamma \in \{m_\Psi\} \;\&\; \Phi(\gamma) \in \{n_\Psi\} \Rightarrow \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A_{\alpha,\mathbf{y}}} \Phi, \lambda\gamma . y).$$

Given $\delta \in V_0$, take any $\Psi_0 \in C(N^N)$ with $\Psi_0(\beta) = \delta$, and take $z = \max(\mathrm{lh}\, m_{\Psi_0}, \mathrm{lh}\, n_{\Psi_0})$ and $y = y_{\Psi_0}$. Then we have

$$(\gamma \in V_0)(\Phi)(\bar{\gamma} z = \bar{\beta} z \;\&\; [\overline{\Phi(\gamma)}]\, z = \bar{\delta} z \Rightarrow \gamma \in [\![\mathbf{A}(\alpha,\mathbf{y})]\!]^{T}_{A_{\alpha,\mathbf{y}}} \Phi, \lambda\gamma . y),$$

which is the hypothesis of 4.3.19 restricted to V_0.

Thus by 4.3.19 we have

$$V_0 \subseteq [\![\exists\tau\forall\alpha\exists\mathbf{y}(\tau(\bar{\alpha}\mathbf{y}) > 0 \wedge \forall\mathbf{x}(\tau(\bar{\alpha}\mathbf{x}) > 0 \rightarrow \mathbf{x} = \mathbf{y}) \wedge \mathbf{A}(\alpha, \tau(\bar{\alpha}\mathbf{y}) \dot{-} 1))]\!]^{T}_{A},$$

and so $[\![\forall\alpha\exists!\,\mathbf{y}\,\mathbf{A}(\alpha,\mathbf{y}) \rightarrow \exists\tau\forall\alpha\exists\mathbf{y}(\tau(\bar{\alpha}\mathbf{y}) > 0 \wedge \forall\mathbf{x}(\tau(\bar{\alpha}\mathbf{x}) > 0 \rightarrow \mathbf{x} = \mathbf{y}) \wedge \mathbf{A}(\alpha, \tau(\bar{\alpha}\mathbf{y}) \dot{-} 1))]\!]^{T}_{A} = N^N$

as required.

5. FORCING VIA TOPOLOGICAL MODELS

5.0.1. Remark. As we pointed out in Chapter Two, the Intuitionistic interpretation of logic, arithmetic, etc, has to do with some abstract notion of 'proof'. Dana Scott in [14] says that the topological interpretation of H. Rasiowa and R. Sikorski (which we gave in Chapter Three and extended in Chapter Four) is not unrelated to this formalization of the notion of 'proof'.

In fact he says we may view a neighbourhood of a topological space as a kind of 'proof' in the following way. Let X and Y be open subsets of a topological space T; let $t \in T$ be a point and let U be an open neighbourhood of t, i.e. $t \in U \subseteq T$. Then U is a 'proof' that $t \in (X \cap Y)$ iff $U \subseteq (X \cap Y)$, i.e. U is a 'proof' that $t \in X$ and that $t \in Y$. Similarly U is a 'proof' that $t \in (X \cup Y)$ iff U contains a neighbourhood U' of t which is either a 'proof' that $t \in X$ or a 'proof' that $t \in Y$. And so on. These closely fit the ideas of 2.1.3.

M.C. Fitting in [5] shows the connection between the Intuitionistic concept of 'proof' and the concept of forcing as developed by Cohen and others. In view of these two linkups, namely 'proof'-forcing and 'proof'-topological interpretation, we might expect a third linkup, namely forcing-topological interpretation.

In the first section of this chapter we shall give a brief description of forcing, and in the second section we shall show that, for a given topology, the topological valuations give rise to a type of forcing.

5.1. FINITE FORCING AND FINITE FORCING BY STRUCTURES.

5.1.1. Remarks. This brief description of how to develop forcing and its major result is due to J. Hirschfeld and W.H. Wheeler in [6].

Given a language L and a structure M for L, a technique that has wide uses in Model Theory is the creation of a new language $L(M)$, which is obtained from L by adding names for each element of M to L. We may then extend the structure M to a structure M_A of $L(M)$, where each new constant $\mathbf{m}$ of $L(M)$ is assigned the element m of M.

5.1.2. Definition. The (basic) diagram of a structure M, Diag(M), is the set of all atomic and negated atomic sentences (which collectively are called basic sentences) in $L(M)$ which are true in M.

5.1.3. Remark. The usefulness of the idea of diagram is seen in the following proposition, the proof of which may be found in C.C. Chang and H.J. Keisler [2]. Let M,N be models for L, then M is isomorphically embedded in N iff N can be expanded to a model of the diagram of M.

5.1.4. Definition. A theory ζ in L is a set of consistent sentences of L. A model of theory ζ is a model for L in which the set of consistent sentences which is theory ζ hold. The class of models for theory ζ is denoted by $M(\zeta)$.

5.1.5. Definition. Formula **A** is a $\forall$-formula $=_{df}$ in prenex normal form **A** has no quantifier or one $\forall$ quantifier.

The set of all $\forall$-formulas deducible from a theory ζ is denoted by $\zeta_\forall$.

<u>5.1.6. Remark</u>. J. Hirschfeld and W.H. Wheeler in [6] show that a structure is contained in a model of a theory ζ iff it is a model of $\zeta_\forall$, from which we deduce that $M(\zeta_\forall)$ consists of all substructures of models of ζ.

<u>5.1.7. Definition</u>. Let $\mathbf{A}$ be a set of constant symbols, none of which occur in L. $L(\mathbf{A})$ denotes the expanded language with the function and predicate symbols of L and constant symbols of L and $\mathbf{A}$. $L(\mathbf{A})$ is a normal expansion of $L =_{df}$ $\mathbf{A}$ is an infinte set.

Let ζ be a theory in L and let $L(\mathbf{A})$ be a normal expansion of L. A condition in $L(\mathbf{A})$ relative to ζ is a finite set $\mathbf{P}$ of basic sentences of $L(\mathbf{A})$ such that $\zeta \cup \mathbf{P}$ is a consistent set of sentences in $L(\mathbf{A})$.

A condition $\mathbf{P}$ in $L(\mathbf{A})$ relative to ζ forces a sentence $\mathbf{B}$ in $L(\mathbf{A})$, written $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{B}$, $=_{df}$

if $\mathbf{B}$ is atomic then $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{B} \quad =_{df} \mathbf{B} \in \mathbf{P}$,

if $\mathbf{B}$ is $\neg\mathbf{C}$ then $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \neg\mathbf{C} \quad =_{df} (\mathbf{Q} \supseteq \mathbf{P}) \sim (\mathbf{Q} \Vdash_{\zeta,\mathbf{A}} \mathbf{C})$,

if $\mathbf{B}$ is $\mathbf{C} \vee \mathbf{D}$ then $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{C} \vee \mathbf{D} \quad =_{df} \mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{C}$ or $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{D}$,

if $\mathbf{B}$ is $\exists x\, \mathbf{B}x$ then $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \exists x\, \mathbf{B}x \quad =_{df}$ for some closed term $\mathbf{t}$ in $L(\mathbf{A})$ $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{B}\mathbf{t}$

<u>5.1.8. Remark</u>. It should be noted that for classical theories, since $\mathbf{B} \wedge \mathbf{C} =_{df} \neg(\neg\mathbf{B} \vee \neg\mathbf{C})$ and $\forall x\, \mathbf{B}x =_{df} \neg\exists x \neg\mathbf{B}x$, we can extend this definition of forcing to the cases $\mathbf{C} \wedge \mathbf{D}$, $\forall x\, \mathbf{B}x$. In the former we get $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{C} \wedge \mathbf{D}$ iff

$(\mathbb{Q} \supseteq \mathbb{P})(\mathrm{E}\mathbb{R} \supseteq \mathbb{Q})(\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{C}$ and $\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{D})$, using the fact, which is easily proved by induction on the complexity of formulas, if $\mathbb{Q} \subseteq \mathbb{P}$ and $\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{B}$ then $\mathbb{Q} \Vdash_{\zeta,\mathbb{A}} \mathbb{B}$.

However some authors, including J. Hirschfeld and W.H. Wheeler, and M.C. Fitting define $\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{C} \wedge \mathbb{D} =_{df} \mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{C}$ and $\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{D}$, i.e. as the simple forcing condition, and this is what we shall take. Given this simple forcing condition, the v-operator can no longer be interpreted as the classical operator. The converse situation will occur in section two.

<u>5.1.9. Remark</u>. Given this definition of forcing, J. Hirschfeld and W.H. Wheeler in [6] proceed to prove the following lemma, which indicates why we consider $\zeta_{\forall}$.

<u>5.1.10. Lemma</u>. $\mathbb{P}$ is a condition relative to ζ iff it is a condition relative to $\zeta_{\forall}$. Moreover given a sentence $\mathbb{B}$ in $L(\mathbb{A})$, $\mathbb{P} \Vdash_{\zeta,\mathbb{A}} \mathbb{B}$ iff $\mathbb{P} \Vdash_{\zeta_{\forall},\mathbb{A}} \mathbb{B}$.

<u>5.1.11. Remarks</u>. We now give the connection between $L(\mathbb{A})$ and $L(M)$. Let ζ be a theory in L. Let $M \in M(\zeta_{\forall})$. Let $\mathbb{A}$ be an infinte set of new constants such that card$(\mathbb{A}) \geqslant$ card(M). We assign constants from $\mathbb{A}$ to a set of generators of M, so that each member of M is represented by a closed term of $L(\mathbb{A})$. The closed terms are the names used in Diag(M).

Using the fact that M is a substructure of a model of ζ iff each of its finite subsets is consistent with ζ, i.e. is a condition for ζ, and 5.1.6, we can show that $M \in M(\zeta_{\forall})$ is equivalent to the assertion that each finite subset of Diag(M) is a condition in $L(\mathbb{A})$ relative to ζ.

This equivalence gives rise to the following idea of finite forcing by structures.

5.1.12. Definition. The structure $M \in M(\zeta_\forall)$ finitely forces a sentence $\mathbf{B}$ in $L(\mathbf{A})$, written $M \Vdash \mathbf{B}$, $=_{df}$ there is a finite subset $\mathbf{P}$ of Diag(M) such that $\mathbf{P} \Vdash_{\zeta,\mathbf{A}} \mathbf{B}$.

A structure $M \in M(\zeta_\forall)$ is finitely generic for theory $\zeta =_{df}$ for each sentence $\mathbf{B}$ in $L(\mathbf{A})$, $M \models \mathbf{B}$ iff $M \Vdash \mathbf{B}$, where $\models$ is the usual satisfaction predicate.

5.1.13. Remarks. We shall see in section two that we consider structures with diagrams consistent with ζ and this gives rise to finite forcing by structures.

This next proposition, which is proved by J. Hirschfeld and W.H. Wheeler in [6], gives the connection between finite forcing and finite forcing by structures.

5.1.14. Proposition. Given a theory ζ in a countable language L, $L(\mathbf{A})$ the normal expansion of L, with $\mathbf{A}$ countable, and given any condition $\mathbf{P}$ in $L(\mathbf{A})$ relative to ζ, then there is a finitely generic M for ζ with names from $L(\mathbf{A})$ such that $\mathbf{P}$ is contained in Diag(M).

5.1.15. Remark. We shall give an analogue of this result in section two. However we shall not be using the idea of strong forcing. Rather we shall use weak forcing. A condition $\mathbf{P}$ weakly forces sentence $\mathbf{B} =_{df} \mathbf{P} \Vdash_{\zeta,\mathbf{A}} \neg\neg\mathbf{B}$. This definition gives rise to the following inductive definition of weak forcing:

if $\mathbf{B}$ is atomic, $\quad \mathbf{P} \Vdash^{W}_{\zeta,\mathbf{A}} \mathbf{B} \quad =_{df} \; (\mathbf{Q} \supseteq \mathbf{P})(E\mathbf{R} \supseteq \mathbf{Q})(\mathbf{B} \in \mathbf{R})$;

if $\mathbf{B}$ is $\neg\mathbf{C}$, $\quad \mathbf{P} \Vdash^{W}_{\zeta,\mathbf{A}} \neg\mathbf{C} \quad =_{df} \; (\mathbf{Q} \supseteq \mathbf{P}) \sim (\mathbf{Q} \Vdash^{W}_{\zeta,\mathbf{A}} \mathbf{C})$;

if B is $C \wedge D$, $P \Vdash^{W}_{\zeta,A} C \wedge D \ =_{df}\ (P \Vdash^{W}_{\zeta,A} C$ and $P \Vdash^{W}_{\zeta,A} D)$;

if B is $\forall x Bx$, $P \Vdash^{W}_{\zeta,A} \forall x Bx \ =_{df}$ for each closed term t, $P \Vdash^{W}_{\zeta,A} Bt$.

5.1.16. Remarks. For the cases B atomic, B is $\neg C$, this definition follows immediately from the definition of weakly forces. The case B is $C \wedge D$ follows from: even Intuitionistically $\neg\neg(C \wedge D)$ is $\neg(\neg C \vee \neg D)$, and the definition of weakly forces.

The case $\forall x Bx$ needs the following lemma, which is proved by contradiction, for each B, $P \Vdash^{W}_{\zeta,A} \neg B$ iff $P \Vdash_{\zeta,A} \neg B$, and then we apply the definition of weakly forces.

We extend our inductive definition of weak forcing to the cases B is $C \vee D$, B is $\exists x Bx$, by defining $\vee, \exists$ as the classical operators, $C \vee D =_{df} \neg(\neg C \wedge \neg D)$, $\exists x Bx =_{df} \neg\forall x \neg Bx$.

5.2. FORCING VIA TOPOLOGICAL VALUATIONS.

5.2.1. Remarks. Given a second order language L such as in 4.2.2, H.B. Enderton in [3] shows that Lowenheim-Skolem Theorem holds for secondary (in Henkin's terminology) second order structures. In view of this we can restrict attention to structures which have a countable set $M = \{m_0, m_1, \ldots\}$ and a countable set $F = \{\phi_0, \phi_1, \ldots\}$ of functions on elements on M.

Given such a structure M, with countable domains M,F, add individual constants $m_0, m_1, \ldots$ and function constants $\phi_0, \phi_1, \ldots$ to L. Call this extended language $L(M)$.

Consider the following set of atomic sentences, Struct $L(M) =_{df}$

$$\{0 = m_r, \ldots; m_r^+ = m_s, \ldots; m_r + m_s = m_t, \ldots; m_r \times m_s = m_t, \ldots; \phi_r(m_s) = m_t, \ldots\}$$

5.2.2. Remarks. Given a set D, the Cantor Space topology on 2^D (the set of all functions from D into $\{0,1\}$) is the product topology where $\{0,1\}$ has the discrete topology. So the basic open subsets of 2^D are
$\{i \in \{D \to \{0,1\}\} : i(d) = 0\}$, $\{i \in \{D \to \{0,1\}\} : i(d) = 1\}$,
for each $d \in D$.

Like the Baire Space, the Cantor Space is a totally disconnected complete metric space and hence has the Baire Property. Indeed in some of the proofs, showing that the structure of Chapter Four was indeed a model of Intuitionistic second order arithmetic, where we used the total disconnectedness of the Baire Space and the Baire Property, we could have used the Cantor Space as the index set of our topological structure and carried out the same argument. It is, as mentioned before, the naturalness of Baire Space in interpreting our function domain that led to us using the Baire Space. For this development of forcing, however, we find that the Cantor Space is now the more natural space.

5.2.3. Definition. The subspace I of $2^{\mathcal{L}(M)}$ is defined as follows: $i \in I$ iff $i^{-1}[1] \cup \{\neg A : A \in i^{-1}[0]\}$ is consistent.

5.2.4. Remark. The idea of considering this subspace and using the Baire Property to develop forcing is due to K.A. Bowen in [1], though he states that G. Takeuti in an unpublished paper 'Topological spaces and forcing' developed this idea first. A. Mostowski in Constructive Sets with Applications, Amsterdam, North Holland, 1969, Studies in logic and foundations of mathematics, says that C. Ryll-Nardzewski also developed this idea independently of Takeuti.

5.2.5. Remark. It is easy to show that our subspace I is equivalent to the subspace J defined by:

$i \in J$ iff $i(\mathbf{0} = \mathbf{m}_r) = 1$ for exactly one r,

and $i(\mathbf{m}_r^+ = \mathbf{m}_s) = 1$ for exactly one s, for each r,

and $i(\mathbf{m}_r + \mathbf{m}_s = \mathbf{m}_t) = 1$ for exactly one t, for each r,s,

and $i(\mathbf{m}_r \times \mathbf{m}_s = \mathbf{m}_t) = 1$ for exactly one t, for each r,s,

. . . .

and $i(\phi_r(\mathbf{m}_s) = \mathbf{m}_t) = 1$ for exactly one t, for each r,s,

. . . .

5.2.6. Remark. We can relate this idea of Struct $L(M)$ to the true model theoretic idea of structure in the following way. The first equations $\mathbf{0} = \mathbf{m}_r$, $\mathbf{m}_r^+ = \mathbf{m}_s, \ldots$ give conditions on the elements of the domain M, and in fact define the primitive recursive functions in that domain. The second equations $\phi_r(\mathbf{m}_s) = \mathbf{m}_t, \ldots$ give conditions on the functions on elements in domain M.

Thus for each $i \in I$ we define a structure M_i as follows: M_i has domains M,F. The assignment A from individual variables into M and function variables into F are left unspecified. Define a function $[\![\,]\!]_i$ from closed terms of $L(M)$ into M_i by:

$[\![\mathbf{m}_r]\!]_i =_{df} m_r$,

$[\![\mathbf{0}]\!]_i =_{df} m_r$ iff $i(\mathbf{0} = \mathbf{m}_r) = 1$,

$[\![\mathbf{a}^+]\!]_i =_{df} m_s$ iff $[\![\mathbf{a}]\!]_i = m_r$ and $i(\mathbf{m}_r^+ = \mathbf{m}_s) = 1$,

. . . .

$\| \phi_t(\mathbf{a}) \|_i =_{df} m_s$ iff $\| \mathbf{a} \|_i = m_r$ and $i(\phi_t(m_t) = m_s) = 1$,

. . . .

Because closed terms have no variables appearing in them, they are independent of any assignment of variables, i.e. $[\![\mathbf{a}^+]\!]_i$ is sent to the same element of M for each assignment of variables. Thus we can extend $[\![\,]\!]_i$ to a function $[\![\ \ ]\!]_{A,i}$ from the terms into M,F, where $[\![x]\!]_{A,i} = A(x)$, $[\![\mathbf{a}^+]\!]_{A,i} = [\![\mathbf{a}]\!]^+_{A,i}$, etc. This is what we did in Chapter Four. The reason we now look only at the restricted function $[\![\,]\!]_i$, is that for forcing we want only to look at sentences and they contain no free variables.

<u>5.2.7. Definition</u>. Define a predicate $M_i \models$ on sentences of $L(M)$ by:

$M_i \models \mathbf{a} = \mathbf{b} \ =_{df} [\![\mathbf{a}]\!]_i = [\![\mathbf{b}]\!]_i$ for all closed terms $\mathbf{a}$, $\mathbf{b}$,

$M_i \models \neg \mathbf{A} \ =_{df}$ not $M_i \models \mathbf{A}$,

$M_i \models \mathbf{A} \wedge \mathbf{B} \ =_{df} M_i \models \mathbf{A}$ and $M_i \models \mathbf{B}$,

. . . .

$M_i \models \forall x \mathbf{A}x =_{df} M_i \models \mathbf{S}_x^{m_r}(\mathbf{A}x)$ for each $r \in N$,

$M_i \models \forall \alpha \mathbf{A}\alpha =_{df} M_i \models \mathbf{S}_\alpha^{\phi_r}(\mathbf{A}\alpha)$ for each $r \in N$,

. . . .

<u>5.2.8. Remark</u>. The predicate $M \models$ is the usual satisfaction predicate restricted to sentences. In view of 5.2.6, since each $i \in I$ gives rise to structure M_i, we can define a predicate $i \models$ by: $i \models \mathbf{A}$ iff $M_i \models \mathbf{A}$.

<u>5.2.9. Remark</u>. We are able to define the direct power structure M^I as follows:

The domains are M^I, F^I. The assignments A from variables into these domains are unspecified. Define a function from closed terms of $L(M)$ into M^I by:

for each $i \in I$, for each closed term $\mathbf{a}$, $[\![\mathbf{a}]\!](i) =_{df} [\![\mathbf{a}]\!]_i$

Again we use the 'closedness' of the terms $\mathbf{a}$ to extend this function to a function from the terms into M^I, where $[\![\mathbf{x}]\!]_A(i) = A(\mathbf{x})(i)$ etc. Again we look only at the restricted function because we are interested in sentences.

5.2.10. Definition. Define a function $[\![\,]\!]$ from sentences into $P(I)$ by:

$[\![\mathbf{a} = \mathbf{b}]\!] =_{df} \{i \in I : [\![\mathbf{a}]\!](i) = [\![\mathbf{b}]\!](i)\}$ for all closed terms $\mathbf{a}, \mathbf{b}$,

$[\![\neg \mathbf{A}]\!] =_{df} -[\![\mathbf{A}]\!]$

$[\![\mathbf{A} \wedge \mathbf{B}]\!] =_{df} [\![\mathbf{A}]\!] \cap [\![\mathbf{B}]\!]$

. . . .

$[\![\forall \mathbf{x}\, \mathbf{A}\mathbf{x}]\!] =_{df} \cap_r [\![\mathbf{S}_{\mathbf{x}}^{\mathbf{m}_r}(\mathbf{A}\mathbf{x})]\!]$

$[\![\forall \alpha \mathbf{A}\alpha]\!] =_{df} \cap_r [\![\mathbf{S}_{\alpha}^{\phi_r}(\mathbf{A}\alpha)]\!]$

. . . .

5.2.11. Proposition. $i \in [\![\mathbf{A}]\!]$ iff $i \models \mathbf{A}$

Proof. By induction on complexity of terms. Similar to that in 4.1.3.

5.2.12. Proposition. If M has the discrete topology, then for each closed term $\mathbf{a}$, $[\![\mathbf{a}]\!]$ is continuous in M^I (with the induced product topology).

Proof. By induction on complexity of terms. Similar to that in 4.1.6.

5.2.13. Remark. In view of 5.2.12 and

$\{i \in I : [\![\mathbf{a}]\!](i) = [\![\mathbf{b}]\!](i)\} =$

$\bigcup\{\{i \in I : [\![\mathbf{a}]\!](i) = m_r\} \cap \{i \in I : [\![\mathbf{b}]\!](i) = m_r\} : r \in N\}$,

$[\![\mathbf{a} = \mathbf{b}]\!]$ is open. This together with the property of the interior operator ensures this next definition is well-defined.

5.2.14. Definition. Define a function $[\![\ ]\!]^T$ from sentences into the set of open sets of I by:

$[\![\mathbf{a} = \mathbf{b}]\!]^T =_{df} \{i \in I : [\![\mathbf{a}]\!](i) = [\![\mathbf{b}]\!](i)\}$ for all closed terms $\mathbf{a}, \mathbf{b}$,

$[\![\neg \mathbf{A}]\!]^T =_{df} \mathrm{int}(-[\![\mathbf{A}]\!]^T)$

$[\![\mathbf{A} \wedge \mathbf{B}]\!]^T =_{df} [\![\mathbf{A}]\!]^T \cap [\![\mathbf{B}]\!]^T$

. . . .

$[\![\forall x \mathbf{A}x]\!]^T =_{df} \mathrm{int} \cap_r [\![\mathbf{S}_x^{m_r}(\mathbf{A}x)]\!]^T$

$[\![\forall \alpha \mathbf{A}\alpha]\!]^T =_{df} \mathrm{int} \cap_r [\![\mathbf{S}_\alpha^{\phi_r}(\mathbf{A}\alpha)]\!]^T$

. . . . for all sentences $\mathbf{A}, \mathbf{B}$, etc.

5.2.15. Definition. We use $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ for finite subsets of members of Struct $L(M)$, and negations of members of Struct $L(M)$. These are called conditions. We use $\{p\}$, $\{q\}$, $\{r\}$ for the corresponding neighbourhoods of $2^{\mathrm{Struct}\,L(M)}$ or I.

So if $\mathbf{A} \in \mathfrak{p}$ then $i \in \{p\}$ iff $i(\mathbf{A}) = 1$, if $\neg\mathbf{A} \in \mathfrak{p}$ then $i \in \{p\}$ iff $i(\mathbf{A}) = 0$.

5.2.16. Definition. Define predicates $i \Vdash$ on sentences by:

$i \Vdash \mathbf{A} =_{df} i \in [\![\mathbf{A}]\!]^T$ for each $i \in I$, each sentence $\mathbf{A}$.

Define predicates $\mathfrak{p} \Vdash$ on sentences by:

$\mathfrak{p} \Vdash \mathbf{A} =_{df} \{p\} \subseteq [\![\mathbf{A}]\!]^T$ for each $\mathfrak{p}$, each sentence $\mathbf{A}$.

5.2.17. Remarks. Of course we see that $i\Vdash$ with respect to topological valuation is the analogue of $i\models$ with respect to valuations.

We shall now show that $p\Vdash$ satisfies the inductive definition of weak forcing. However since our conditions include only some atomic sentences (indeed only those with which we can define a structure), we must find an alternative to the clause for A atomic, $p\Vdash A$ iff $(q \supseteq p)(Er \supseteq q)(A \in r)$.

The following result gives a condition which is equivalent to this one in the usual context, but is also appropriate in our context.

5.2.18. Proposition. Given a condition P in $L(A)$ relative to ζ and an atomic sentence B, $P \Vdash^{w}_{\zeta,A} B$ iff $\vdash (\bigwedge P) \rightarrow B$, where $\bigwedge P$ is the conjunction of members of P.

Proof. $\Rightarrow$ we prove the contrapositive

not $\vdash (\bigwedge P) \rightarrow B$ implies not $\vdash \neg((\bigwedge P) \wedge (\neg B))$

implies $P \cup \{\neg B\}$ is consistent

implies $P \cup \{\neg B\}$ is a condition Q such that

$Q \supseteq P$ and $(R \supseteq Q)$ not $(B \in R)$:

[otherwise such an R would be inconsistent]

implies $(EQ \supseteq P)(R \supseteq Q)$ not $(B \in R)$

implies not $(Q \supseteq P)(ER \supseteq Q)(B \in R)$

implies not $P \Vdash^{w}_{\zeta,A} B$.

$\Leftarrow$ If $\vdash (\bigwedge P) \rightarrow B$ then: for some $Q \subseteq P$, $Q \cup \{B\}$ is inconsistent

implies $\vdash (\bigwedge P) \rightarrow B$ and $\vdash \neg((\bigwedge Q) \wedge B)$

implies $\vdash (\bigwedge P) \rightarrow B$ and $\vdash (\bigwedge Q) \rightarrow (\neg B)$

implies $\vdash (\bigwedge P) \to B$ and $\vdash (\bigwedge Q) \to (\neg B)$

and $\vdash (\bigwedge Q) \to (\bigwedge P)$

[since $Q \supseteq P$]

implies $\vdash (\bigwedge Q) \to B$ and $\vdash (\bigwedge Q) \to (\neg B)$

[using modus ponens]

implies $\vdash \neg(\bigwedge Q)$

[using $\vdash (B \to A) \to ((B \to (\neg A)) \to (\neg B))$

and modus ponens].

This contradicts Q being consistent.

Therefore $\vdash (\bigwedge P) \to B$ implies $(Q \supseteq P)(Q \cup \{B\})$ is consistent

implies $(Q \supseteq P)(ER \supseteq Q)(B \in R)$

[taking R to be $Q \cup \{B\}$]

implies $P \Vdash^{w}_{\zeta, A} B$.

5.2.19. Proposition. Using the equivalence of 5.2.18, $p \Vdash$ satisfies the inductive definition of weak forcing.

Proof. If A is atomic then

$\vdash (\bigwedge p) \to A$ iff for each countable structure M, $M \models (\bigwedge p) \to A$

[in view of the Lowenheim-Skolem Theorem]

iff for each countable structure M, if $M \models \bigwedge p$ then $M \models A$

iff for each $i \in I$, if $i \models \bigwedge p$ then $i \models A$

[By 5.2.6 each $i \in I$ gives rise to a countable structure M_i; other countable structures differ from these M_i only in their domains, and we can find 1-1 onto functions between these domains.]

iff $[\![\bigwedge p]\!] \subseteq [\![A]\!]$

[by 5.2.11]

iff $\{p\} \subseteq [\![A]\!]^{T}$

[since $[\![A]\!]^{T} = [\![A]\!]$ for A atomic and

$[\![\bigwedge p]\!] = \cap\{[\![B]\!] : B \in p\} = \cap\{[\![B]\!]^T : B \in p\} = [\![\bigwedge p]\!]^T = \{p\}$].

If A is $\neg B$, $p \Vdash \neg B$ iff $\{p\} \subseteq [\![\neg B]\!]^T$

iff $\{p\} \subseteq \text{int} -[\![B]\!]^T$

iff $\{p\} \subseteq -[\![B]\!]^T$

[since $\{p\}$ is open, $\Leftarrow$) holds].

$\{p\} \subseteq -[\![B]\!]^T$ implies not $\{p\} \subseteq [\![B]\!]^T$

implies $(\{q\} \subseteq \{p\})(\text{not } \{q\} \subseteq [\![B]\!]^T)$.

If $(\{q\} \subseteq \{p\})(\text{not } \{q\} \subseteq [\![B]\!]^T)$: not $\{p\} \subseteq -[\![B]\!]^T$

implies $\{p\} \cap [\![B]\!]^T$ is a non-empty open set

implies there is a neighbourhood $\{q\}$ with

$\{q\} \subseteq \{p\} \cap [\![B]\!]^T$

implies there is a neighbourhood $\{q\}$ with

$\{q\} \subseteq \{p\}$ and $\{q\} \subseteq [\![B]\!]^T$.

This contradicts $(\{q\} \subseteq \{p\})(\text{not } \{q\} \subseteq [\![B]\!]^T)$.

Therefore $\{p\} \subseteq -[\![B]\!]$ iff $(\{q\} \subseteq \{p\})(\text{not } \{q\} \subseteq [\![B]\!]^T)$

iff $(q \geqslant p) \sim (q \Vdash B)$

[where $\geqslant$ is set inclusion].

If A is $B \wedge C$ then

$p \Vdash B \wedge C$ iff $\{p\} \subseteq [\![B \wedge C]\!]^T$

iff $\{p\} \subseteq [\![B]\!]^T \cap [\![C]\!]^T$

iff $\{p\} \subseteq [\![B]\!]^T$ and $\{p\} \subseteq [\![C]\!]^T$

iff $p \Vdash B$ and $p \Vdash C$.

If A is $\forall x Ax$ then

$p \Vdash \forall x Ax$ iff $\{p\} \subseteq [\![\forall x Ax]\!]^T$

iff $\{p\} \subseteq \text{int} \cap_r [\![S_x^{m_r}(Ax)]\!]^T$

iff $\{p\} \subseteq \cap_r [\![S_x^{m_r}(Ax)]\!]^T$

[since $\{p\}$ is open, $\Leftarrow$) holds]

iff $\{p\} \subseteq [\![S_x^{m_r}(Ax)]\!]^T$ for each r

iff (closed term $\mathbf{m}_r$ in $L(M)$) $(\mathfrak{p} \Vdash S_x^{\mathbf{m}_r}(\mathbf{A}x))$

iff (closed term $\mathbf{m}_r$ in $L(M)$)$(\mathfrak{p} \Vdash \mathbf{A}\mathbf{m}_r)$.

Similarly for $\mathbf{A}$ is $\forall\alpha\mathbf{A}\alpha$.

5.2.20 Remarks. Because we can not show $\{p\} \subseteq [\![\mathbf{B} \vee \mathbf{C}]\!]^T$ implies $(\{q\} \subseteq \{p\})(E\{r\} \subseteq \{q\})(\{r\} \subseteq [\![\mathbf{B}]\!]^T$ or $\{r\} \subseteq [\![\mathbf{C}]\!]^T)$,

nor the converse, we are forced to define $\vee$ as the classical operator, that is $\mathbf{B} \vee \mathbf{C} =_{df} \neg(\neg\mathbf{B} \wedge \neg\mathbf{C})$. Then as in 5.1.17 we can show that $\mathfrak{p} \Vdash$ satisfies the inductive definition of weak forcing. Similarly for $\exists x$, $\exists\alpha$.

5.2.19 establishes the connection between $\mathfrak{p} \Vdash$ and $\mathfrak{p} \Vdash^{w}_{\zeta,\mathbf{A}}$, for the case when ζ is empty and $\mathbf{A}$ is $\{\mathbf{m}_0, \mathbf{m}_1, \ldots \phi_0, \phi_1, \ldots\}$. The fact that is not obvious is that our conditions $\mathfrak{p}$ are consistent subsets of basic sentences. However for the predicate $\mathfrak{p} \Vdash$ we only consider those $\mathfrak{p}$ whose $\{p\} \subseteq P(I)$. If $\mathfrak{p}$ is inconsistent then for some $\mathbf{A}$, $\vdash (\bigwedge\mathfrak{p}) \to \mathbf{A}$ and $\vdash (\bigwedge\mathfrak{p}) \to (\neg\mathbf{A})$. But as we saw in the proof of 5.2.19, this implies $[\![\bigwedge\mathfrak{p}]\!] \subseteq [\![\mathbf{A}]\!]$ and $[\![\bigwedge\mathfrak{p}]\!] \subseteq [\![\neg\mathbf{A}]\!] = -[\![\mathbf{A}]\!]$, a contradiction.

We can easily extend this connection between $\mathfrak{p} \Vdash$ and $\mathfrak{p} \Vdash^{w}_{\zeta,\mathbf{A}}$ for ζ not empty, by considering the subspace J of I given by: $i \in J$ iff $\zeta \cup i^{-1}[1] \cup \{\,\mathbf{A} : \mathbf{A} \in i^{-1}[0]\}$ is consistent; and defining conditions $\mathfrak{p}$ and predicates $\mathfrak{p} \Vdash$ relative to ζ by: $\mathfrak{p} \Vdash \mathbf{A} =_{df} \{p\} \subseteq [\![\mathbf{A}]\!]^T \cap J$.

Our next task is to establish the connection between $i \Vdash$ and $M \Vdash$. Recall that $i \Vdash$ is really $M_i \Vdash$.

5.2.21. Proposition. For each $i \in I$, $\mathbf{A} \in L(M)$, $i \Vdash \mathbf{A}$ iff for some $\mathfrak{p}$ with $i \in \{p\}$, $\mathfrak{p} \Vdash \mathbf{A}$.

Proof. $i \Vdash A$ iff $i \in \llbracket A \rrbracket^T$

iff for some $\{p\}$, $i \in \{p\}$ and $\{p\} \subseteq \llbracket A \rrbracket^T$

[since $\llbracket A \rrbracket^T$ is open and the neighbourhoods of I are $\{p\}$ for some $\mathfrak{p}$]

iff for some $\mathfrak{p}$ with $i \in \{p\}$, $\mathfrak{p} \Vdash A$.

5.2.22. Remark. 5.2.21 says $M_i \Vdash A$ iff $(E\mathfrak{p})(i \in \{p\}$ and $\mathfrak{p} \Vdash A)$.

Now $i \in \{p\}$ iff $\{A : A \in \mathfrak{p}\} \subseteq \{A : i(A) = 1\}$ and

$\{\neg A : \neg A \in \mathfrak{p}\} \subseteq \{\neg A : i(A) = 0\}$

iff $\{A : A \in \mathfrak{p}\} \cup \{\neg A : \neg A \in \mathfrak{p}\} \subseteq$

$\{A : i(A) = 1\} \cup \{\neg A : i(A) = 0\}$

[since $\{A : A \in \mathfrak{p}\}$, $\{\neg A : \neg A \in \mathfrak{p}\}$, $\{A : i(A) = 1\}$, $\{\neg A : i(A) = 0\}$ are disjoint]

iff $\mathfrak{p} \subseteq \{A : i(A) = 1\} \cup \{\neg A : i(A) = 0\}$.

Now since $A \in \text{Struct}\, L(M)$, by the definition in 5.2.6,

$i \in \llbracket A \rrbracket$ iff $i(A) = 1$, and so $i \in \llbracket \neg A \rrbracket$ iff $i(A) = 0$.

Therefore $i \in \{p\}$ iff $\mathfrak{p} \subseteq \{A : A \in \text{Struct}\, L(M)$ and $i \in \llbracket A \rrbracket\} \cup$

$\{\neg A : A \in \text{Struct}\, L(M)$ and $i \in \llbracket \neg A \rrbracket\}$

iff $\mathfrak{p} \subseteq \{A : A$ is atomic and $M_i \models A\} \cup$

$\{\neg A : A$ is atomic and $M_i \models \neg A\}$

iff $\mathfrak{p} \subseteq \text{Diag}(M_i)$.

Hence 5.2.23 says $M_i \Vdash A$ iff $(E\mathfrak{p})(\mathfrak{p} \subseteq \text{Diag}(M_i)$ and $\mathfrak{p} \Vdash A)$, establishing the connection between $M_i \Vdash$ and $M \Vdash$ of section one.

Notice this connection depends on our using $\llbracket\ \rrbracket^T$ rather than $\llbracket\ \rrbracket$.

5.2.23. Definition. $i \in I$ is generic $=_{df}$ $i \Vdash A$ iff $i \models A$ for each sentence A.

<u>5.2.24. Proposition.</u> i is generic iff $i \in (\llbracket A \rrbracket^T \cup \llbracket \neg A \rrbracket^T)$ for each sentence A.

<u>Proof.</u> $\Rightarrow$) For each sentence A either $i \models A$ or not $i \models A$

implies either $i \models A$ or $i \models \neg A$

implies either $i \Vdash A$ or $i \Vdash \neg A$

[since i is generic]

implies either $i \in \llbracket A \rrbracket^T$ or $i \in \llbracket \neg A \rrbracket^T$

implies $i \in (\llbracket A \rrbracket^T \cup \llbracket \neg A \rrbracket^T)$.

$\Leftarrow$) Firstly, using the properties of our interior operator, $\llbracket A \rrbracket^T$ and $\llbracket \neg A \rrbracket^T$ are disjoint, so that

$i \in (\llbracket A \rrbracket^T \cup \llbracket \neg A \rrbracket^T)$ implies $i \in \llbracket A \rrbracket^T$ iff not $i \in \llbracket \neg A \rrbracket^T$. (*)

Then we show $\Leftarrow$) by assuming that for each formula A, $i \in (\llbracket A \rrbracket^T \cup \llbracket \neg A \rrbracket^T)$, and proving that $i \Vdash A$ iff $i \models A$ by induction on the complexity of A.

For $a = b$ we have $i \Vdash a = b$ iff $i \models a = b$ by definition.

Now supposing that $i \Vdash A$ iff $i \models A$,

for $\neg A$, $i \Vdash \neg A$ iff not $i \Vdash A$

[by (*)]

iff not $i \models A$

[by hypothesis]

iff $i \models \neg A$.

For $A \wedge B$, $i \Vdash A \wedge B$ iff $i \Vdash A$ and $i \Vdash B$

iff $i \models A$ and $i \models B$

[by hypothesis]

iff $i \models A \wedge B$.

For $\forall x Ax$, $i \Vdash \forall x Ax$ iff for each r, $i \Vdash S_x^{m_r}(Ax)$

iff for each r, $i \models S_x^{m_r}(Ax)$

[by hypothesis]

iff $i \models \forall x Ax$

Similarly for $\forall\alpha A\alpha$, etc.

5.2.25. Remark. For any sentence A, $[\![A]\!]^T$, $[\![\neg A]\!]^T$ are open sets so that the complement of their union is closed. Hence

$$\begin{aligned}\text{int cl} - ([\![A]\!]^T \cup [\![\neg A]\!]^T) &= \text{int} - ([\![A]\!]^T \cup [\![\neg A]\!]^T)\\ &= -\,\text{cl}\,([\![A]\!]^T \cup [\![\neg A]\!]^T)\\ &= -\,(\,\text{cl}\,[\![A]\!]^T \cup \text{cl}\,[\![\neg A]\!]^T)\\ &= -\,(\,\text{cl}\,[\![A]\!]^T \cup \text{cl int} - [\![A]\!]^T)\\ &= -\,(\,\text{cl}\,[\![A]\!]^T \cup - \text{int cl}\,[\![A]\!]^T)\end{aligned}$$

[and since $\text{int cl}\,[\![A]\!]^T \subseteq \text{cl}\,[\![A]\!]^T$] $= -I = \phi$.

Thus for each sentence A, $-([\![A]\!]^T \cup [\![\neg A]\!]^T)$ is a nowhere dense set. Therefore $\cap\{([\![A]\!]^T \cup [\![\neg A]\!]^T) : A \in L(M)\}$ is a co-meagre set and therefore a co-boundary set in I. (This depends on the fact that any subspace of a space having the Baire Property has the Baire Property, and also that $L(M)$ is a countable language.)

Therefore in view of 5.2.24, the set of generic points is a co-boundary set, and so generic points do exist.

5.2.26. Remark. In the same manner as in 5.2.20, we can extend the connection between $i\Vdash$ and $M\Vdash$ to models of theories ζ which are non-empty, by considering the subspace J of I given by: $i \in J$ iff $\zeta \cup i^{-1}[1] \cup \{\neg A : A \in i^{-1}[0]\}$ is consistent. Again we use the fact that a subspace of a space having the Baire Property has the Baire Property to set up the definition, and facts about generic $i \in J$.

We are now able to give the analogue of 5.1.14, provided that we associate a condition $\mathfrak{p}$ relative to a theory ζ with the non-empty neighbourhood $\{p\} \cap J$ of J defined above. For if $\mathfrak{p}$ is a condition relative to ζ then

$\zeta \cup \mathfrak{p}$ is consistent, and so has a countable model by the Löwenheim-Skolem Theorem for second order logic. Such a model is isomorphic to M_i for some $i \in I$, and here $i \in \{p\} \cap J$.

5.2.27. Proposition. Given a theory ζ in $L(M)$ and given a condition $\mathfrak{p}$ relative to ζ, then there is a generic M_i for ζ such that $\mathfrak{p} \subseteq \mathrm{Diag}(M_i)$.

Proof. Given a condition $\mathfrak{p}$ relative to ζ, since the set of generic points in J is dense, the set of generic points in J intersects with the neighbourhood $\{p\}$ non-emptily. Therefore there is a generic $i \in I$ such that $i \in \{p\}$.

We saw in 5.2.22 that $i \in \{p\}$ iff $\mathfrak{p} \subseteq \mathrm{Diag}(M_i)$

Therefore there is a generic M_i for ζ (since $i \in J$) such that $\mathfrak{p} \subseteq \mathrm{Diag}(M_i)$.

5.2.28. Remark. The model M_i associated with the generic point $i \in \{p\}$ satisfies ζ and each formula forced by ζ.

We have seen the power of using topology: using the Baire Property we have shown that generic points (or structures) exist and that an analogue of 5.1.14 holds. The usual proofs of these are considerably longer.

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